Single Pass Spectral Sparsification in Dynamic Streams

2014.10.21

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Massachusetts Institute of Technology

1-Pass Spectral Sparsification in Dynamic Streams

Overview

- In $\tilde{O}(n)$ space, maintain a graph compression from which we can always return a spectral sparsifier.

Main technique

- Use ℓ_2 heavy hitter sketches to sample by effective resistance in the streaming model.

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Outline

1. Graph Sparsification
2. Semi-Streaming Computational Model
3. Prior Work Review
4. Our Algorithm
   - Sampling in the Streaming Model
   - Recursive Sparsification [Li, Miller, Peng '12]

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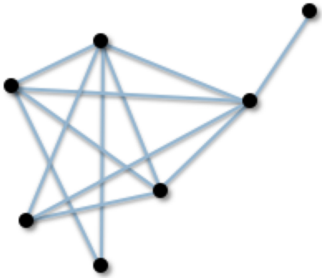
Graph Sparsification

General Idea

- Approximate a dense graph with a much sparser graph.
- Reduce $O(n^2)$ edges $\rightarrow$ $O(n \log n)$ edges



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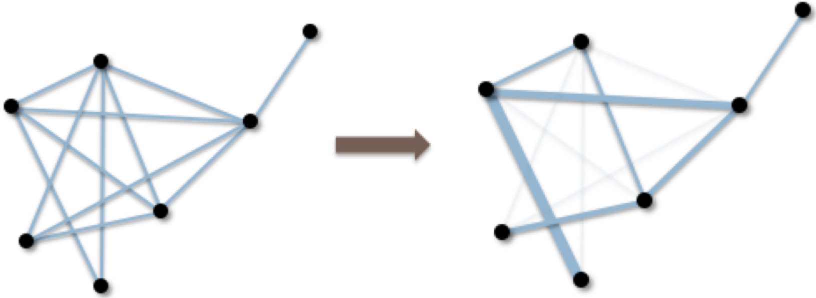
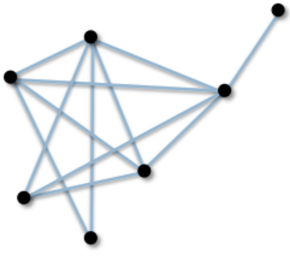


Graph Sparsification

Cut Sparsification (Benczúr, Karger '96)

- Preserve *every* cut value to within $(1 \pm \varepsilon)$ factor



Applications: Minimum cut, sparsest cut, etc.

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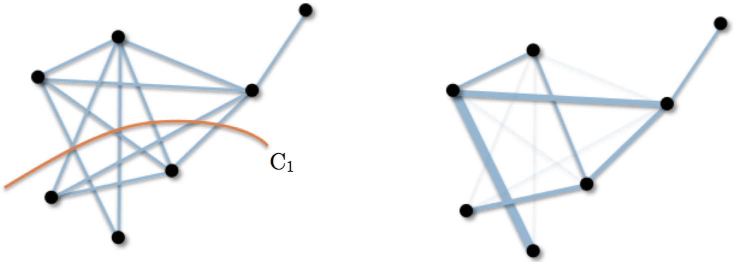

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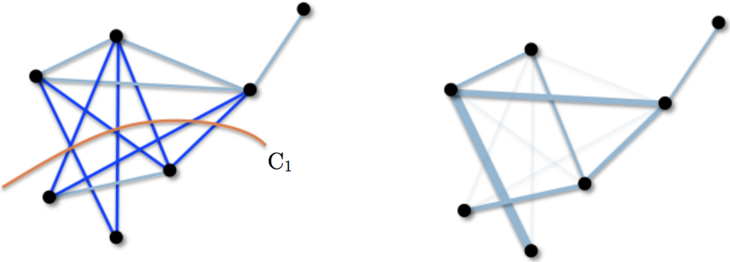

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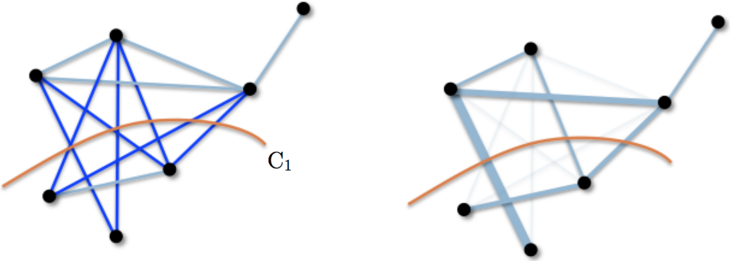

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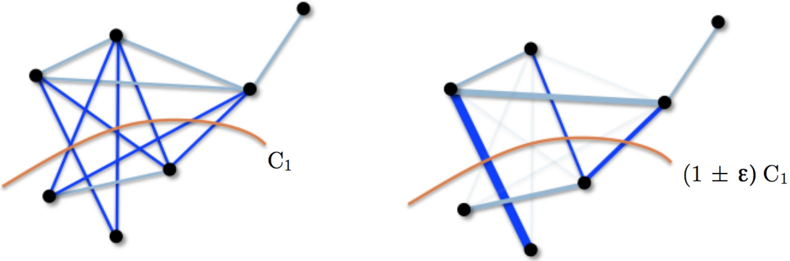

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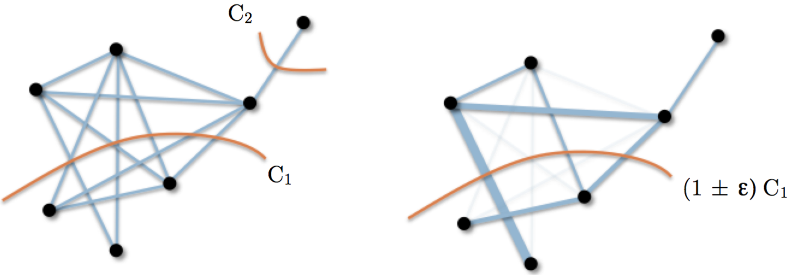

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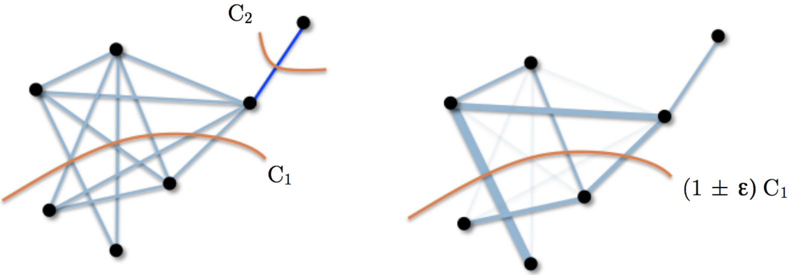

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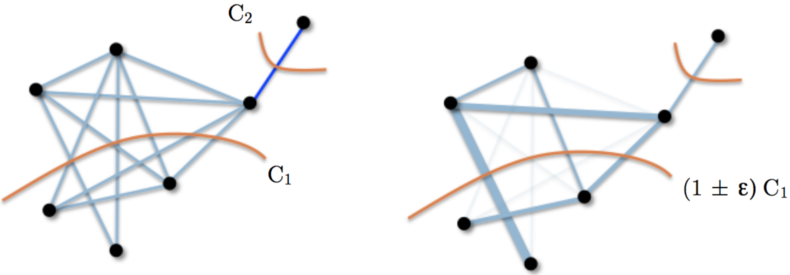

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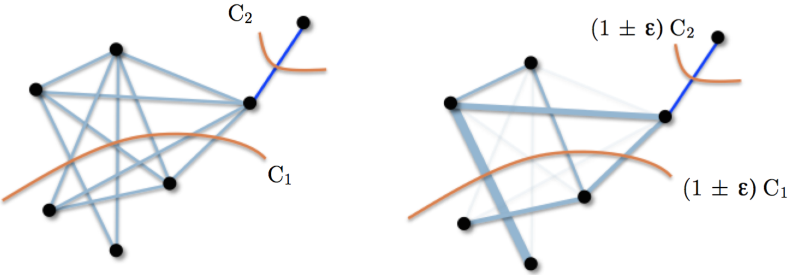

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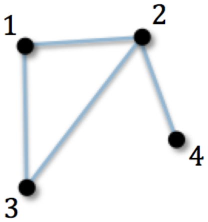

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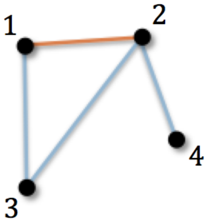

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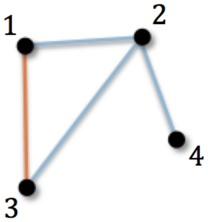

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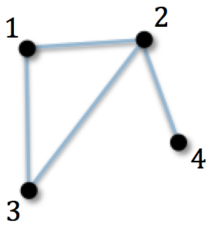

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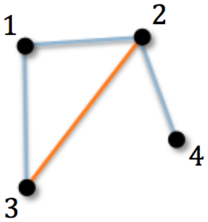

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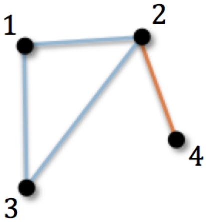

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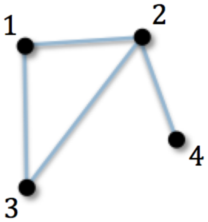

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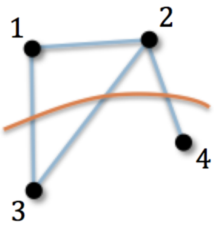

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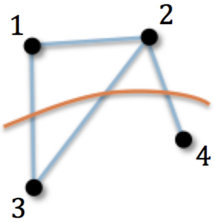

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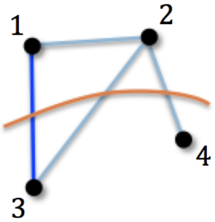

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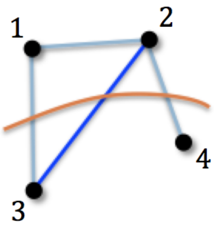

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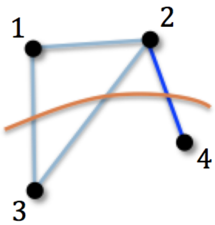

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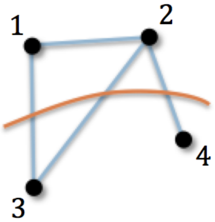

$\|\mathbf{B}\mathbf{x}\|_2^2 = \text{cut value}$

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Goal

Find some $\tilde{\mathbf{B}}$ such that, for all $\mathbf{x} \in \{0,1\}^n$,

$$(1-\varepsilon)\|\mathbf{B}\mathbf{x}\|_2^2 \leq \|\tilde{\mathbf{B}}\mathbf{x}\|_2^2 \leq (1+\varepsilon)\|\mathbf{B}\mathbf{x}\|_2^2$$

- $\mathbf{x}^\top \tilde{\mathbf{B}}^\top \tilde{\mathbf{B}}\mathbf{x} \approx \mathbf{x}^\top \mathbf{B}^\top \mathbf{B}\mathbf{x}$.
- $\mathbf{L} = \mathbf{B}^\top \mathbf{B}$ is the *graph Laplacian*.

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Spectral Sparsification (Spielman, Teng '04)

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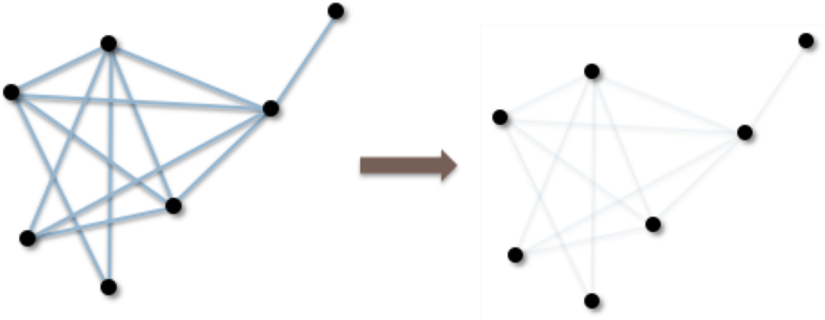
How are sparsifiers constructed?

Randomly sample edges (i.e. rows from $\mathbf{B}$):



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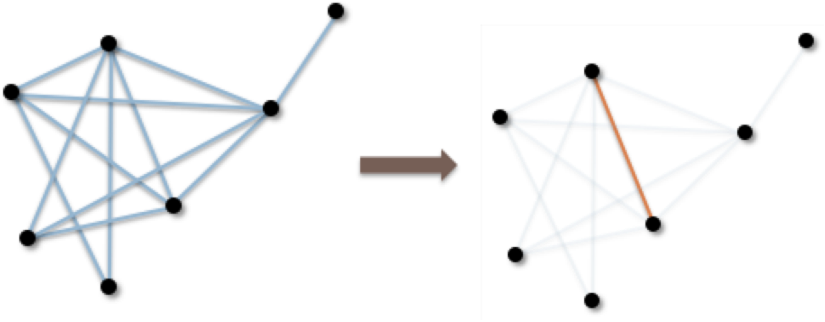

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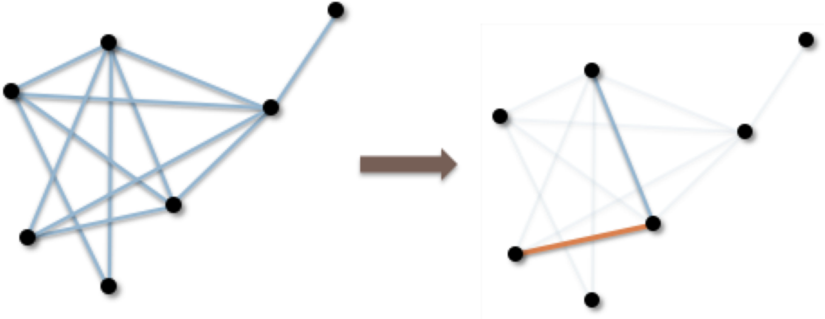

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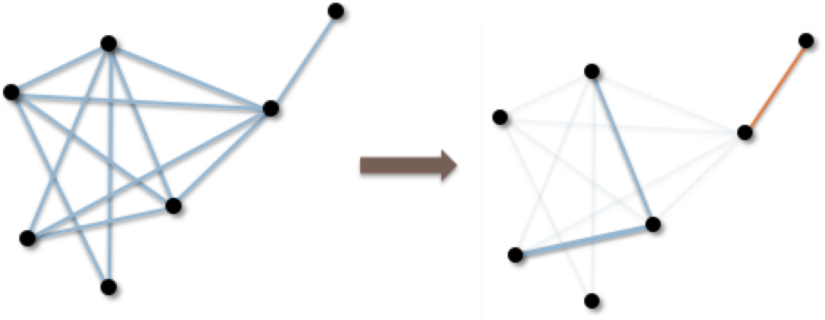

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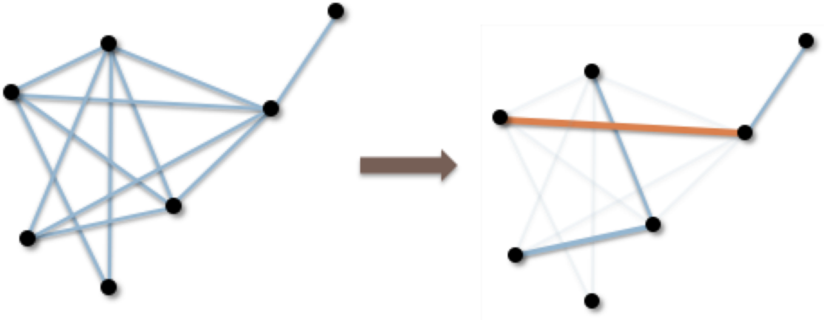

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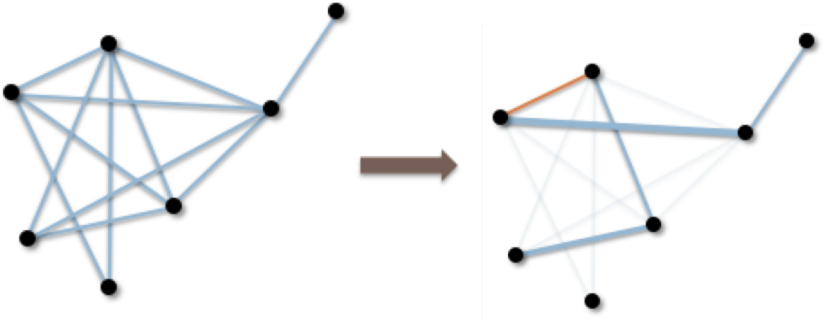

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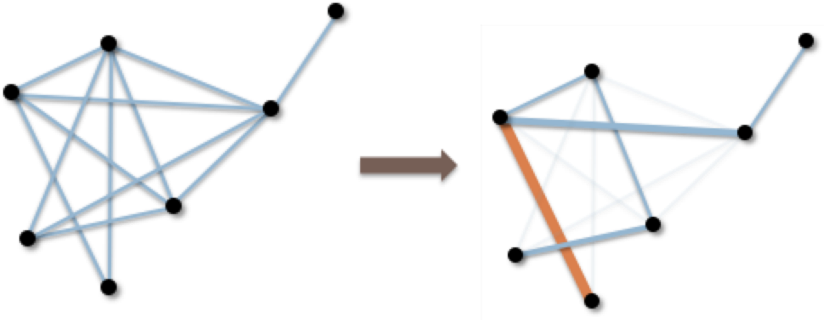

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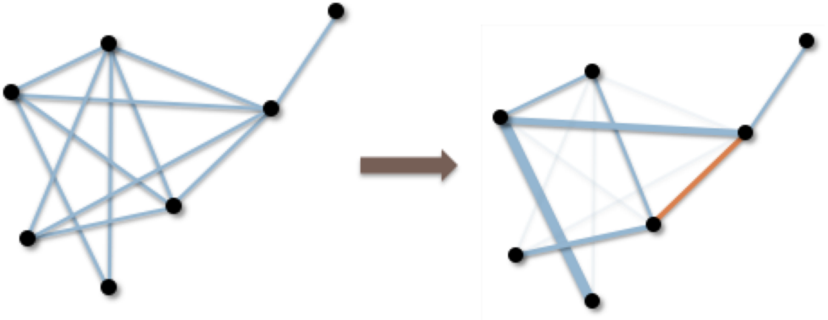

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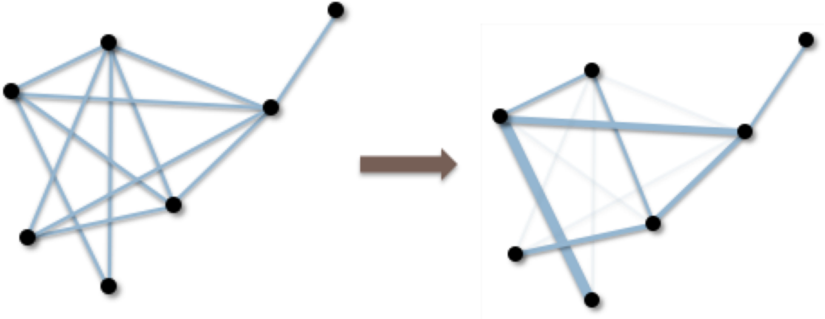

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Motivation

- Makes sense to compress a graph, but what if we cannot afford to store it in the first place?
- Is it possible to "sketch" a graph in small space by maintaining a sparsifier or some other representation?

Overview

Semi-Streaming Model

Introduced by Feigenbaum, Kannan, McGregor, Suri, Zhang '05.

- Space allowance $n \log^c(n)$.
- Receive data via edge updates.
- Minimum spanning tree, maximal matching, graph connectivity, etc.

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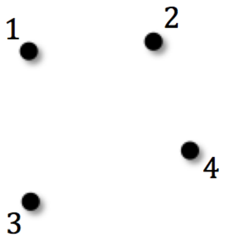

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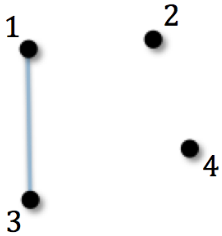

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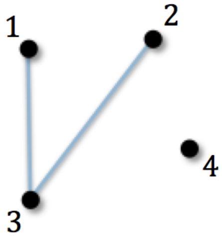

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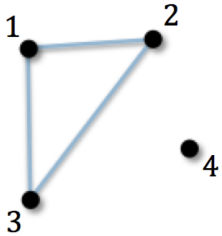

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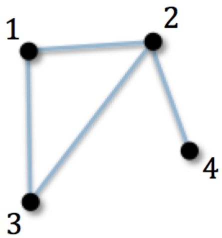

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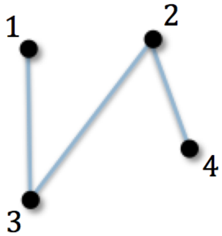

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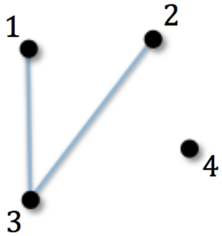

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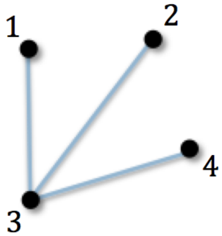

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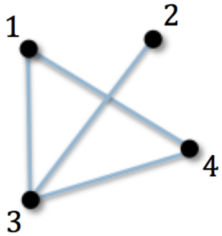

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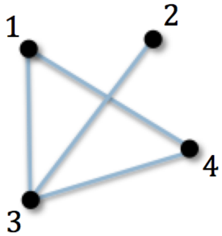

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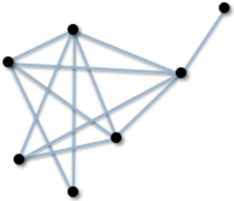

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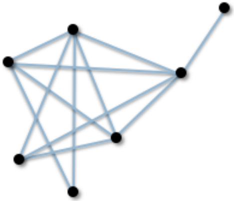

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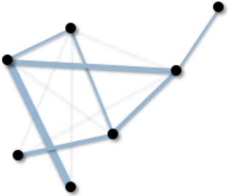

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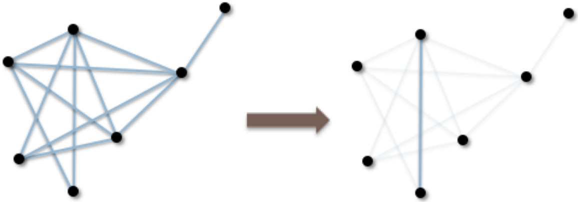

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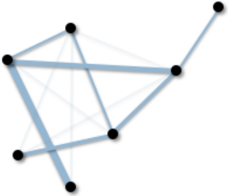

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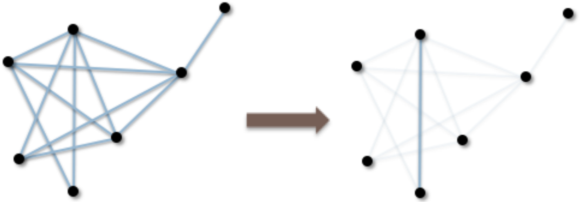

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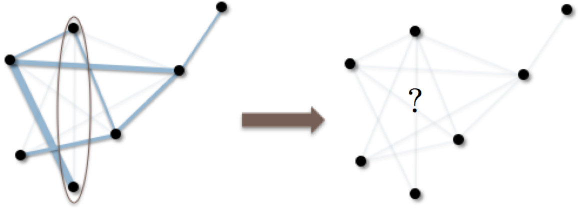

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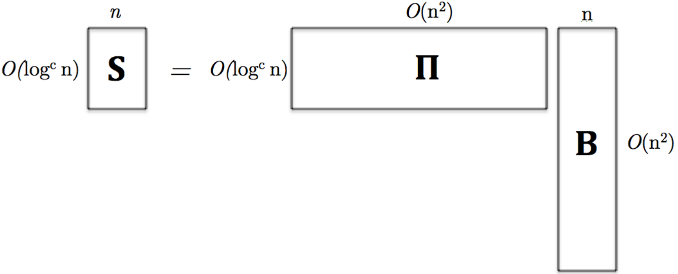

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We are still going to sample by effective resistance.

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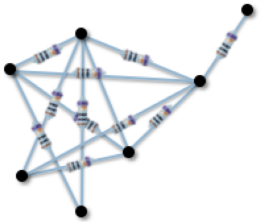

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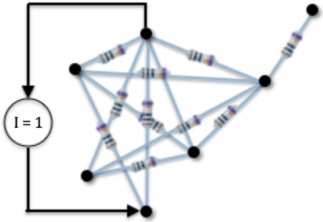

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Effective resistance of edge e is $\tau_e = \mathbf{x}_e^\top \mathbf{L}^{-1} \mathbf{x}_e$.

Alternatively, τ_e is the e^{th} entry in the vector:

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We just need two more ingredients: $\mathbf{B}\mathbf{L}^{-1}\mathbf{x}_e$

ℓ_2 **Heavy Hitters [GLPS10]:**

- Sketch vector *poly*(n) vector in *polylog*(n) space.
- Extract any element who's square is a $O(1/\log n)$ fraction of the vector's squared norm.

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- $\tilde{\mathbf{L}}$ such that $\mathbf{x}^\top\tilde{\mathbf{L}}\mathbf{x} = (1 \pm \mathit{constant})\mathbf{x}^\top\mathbf{L}\mathbf{x}$

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Putting it all together: $\mathbf{B}\mathbf{L}^{-1}\mathbf{x}_e$

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4. Extract heavy hitters from the vector, check if e^{th} entry is one.

$$\frac{\mathbf{B}\tilde{\mathbf{L}}^{-1}\mathbf{x}_e(e)^2}{\|\mathbf{B}\tilde{\mathbf{L}}^{-1}\mathbf{x}_e\|_2^2} \approx \frac{\tau_e^2}{\tau_e} = \tau_e$$

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Putting it all together: $\mathbf{BL}^{-1}\mathbf{x}_e$

1. Sketch $(\mathbf{\Pi}_{\text{heavy hitters}})\mathbf{B}$ in $n \log^c n$ space.
2. Compute $(\mathbf{\Pi}_{\text{heavy hitters}})\mathbf{B}\tilde{\mathbf{L}}^{-\mathbf{1}}$.
3. For every possible edge e, compute $(\mathbf{\Pi}_{\text{heavy hitters}})\mathbf{B}\tilde{\mathbf{L}}^{-\mathbf{1}}\mathbf{x}_e$
4. Extract heavy hitters from the vector, check if e^{th} entry is one.

$$\frac{\mathbf{B}\tilde{\mathbf{L}}^{-1}\mathbf{x}_e(e)^2}{\|\mathbf{B}\tilde{\mathbf{L}}^{-1}\mathbf{x}_e\|_2^2} \approx \frac{\tau_e^2}{\tau_e} = \tau_e$$

So, as long as $\tau_e > O(1/\log n)$, we will recover the edge!

Sampling in the Streaming Model

How about edges with lower effective resistance? Sketch:

$\mathbf{BL}^{-1}\mathbf{x}_e$

Sampling in the Streaming Model

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$\mathbf{B}$

$\mathbf{BL^{-1}x}_e$

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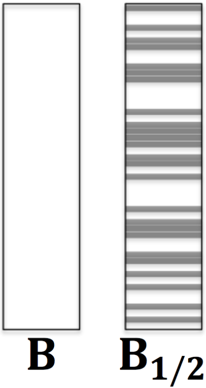

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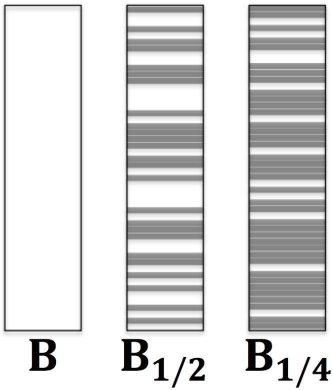

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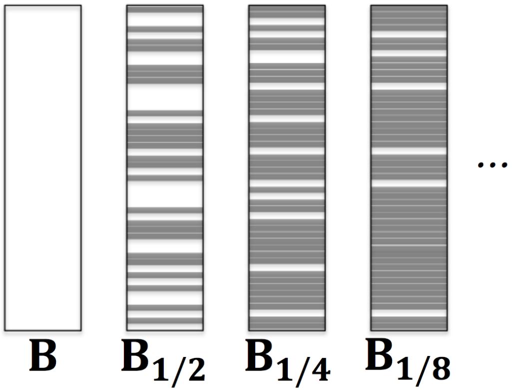

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Sampling in the Streaming Model

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How about edges with lower effective resistance?

- First level: $\tau_e > 1/\log n$ with probability 1.
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So, we can sample every edge by (effective resistance) $\times$ $O(\log n)$.

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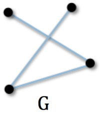
Overview

Sparsifer Chain

Final Piece [Li, Miller, Peng '12]

- We needed a constant error spectral sparsifier to get our $(1 \pm \epsilon)$ sparsifier.



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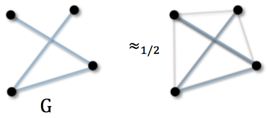

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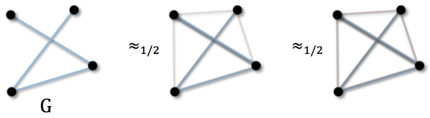

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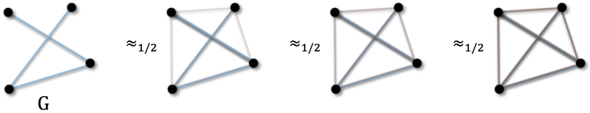

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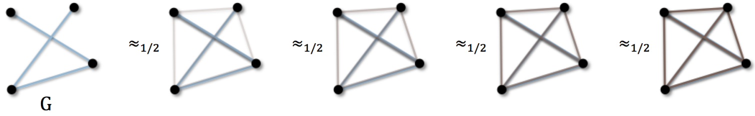

Conclusion

Final Thoughts:

- Note that everything we did extends unmodified to general matrices $\mathbf{B}$ and general quadratic forms $\mathbf{B}^\top\mathbf{B}$.
  - Just need to ensure that we have a row dictionary and can thus test every possible entry.
- Generically, storing a compression of $\mathbf{B}^\top\mathbf{B}$ takes $\Omega(n^2)$ space. Avoid lower bound simply when the row dictionary is $\mathrm{poly}(n)$ size.

Conclusion

Thank you!