CS-GY 6763: Lecture 12
Stochastic Block Model, subspace embeddings + ϵ-net arguments

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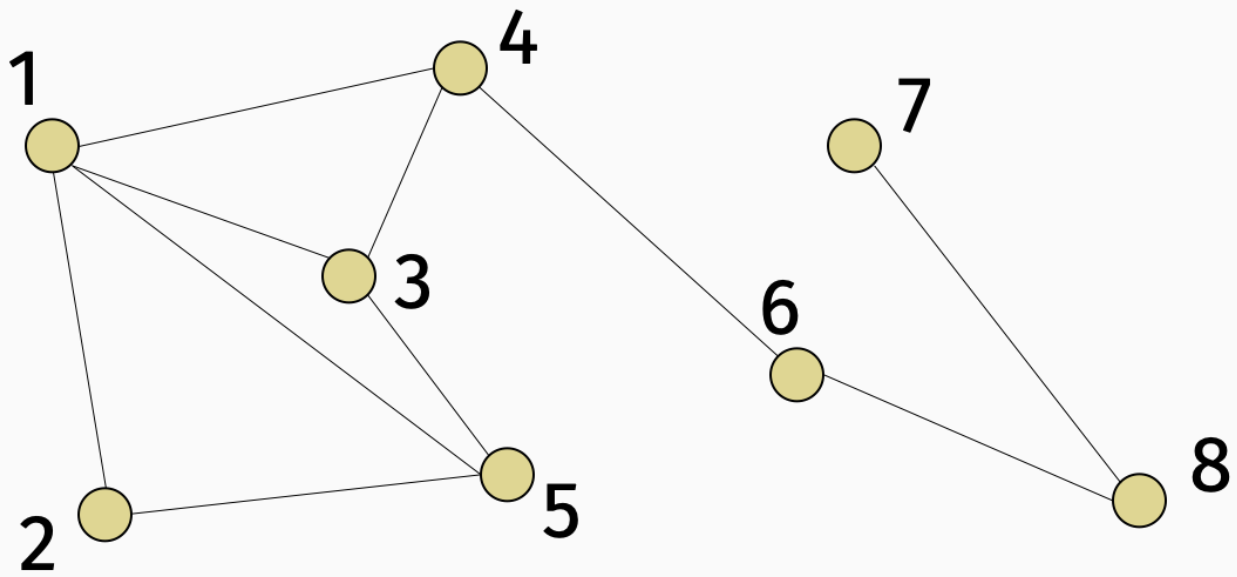
Main idea: Understand graph data by constructing natural matrix representations, and studying that matrix's spectrum (eigenvalues/eigenvectors).

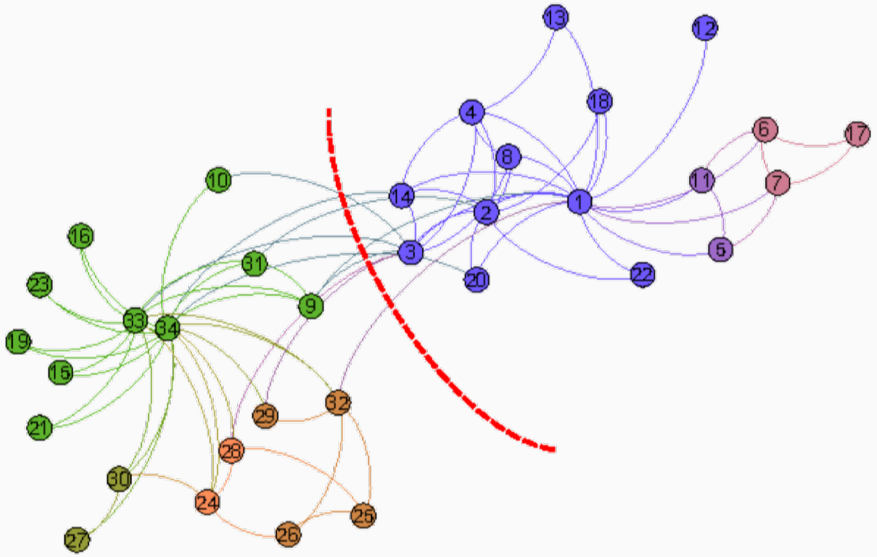


$G = (V, E)$ is an undirected, unweighted graph with n nodes.

BALANCED CUT

Goal: Given a graph $G = (V, E)$, partition nodes along a cut that:

- Has few crossing edges: $|\{(u, v) \in E : u \in S, v \in T\}|$ is small.
- Separates large partitions: $|S|, |T|$ are not too small.



(a) Zachary Karate Club Graph

Applications: Understanding community structure in social networks, partitioning finite element meshes, non-linear clustering in machine learning, data visualization, etc. etc.

β-Balanced Cut:

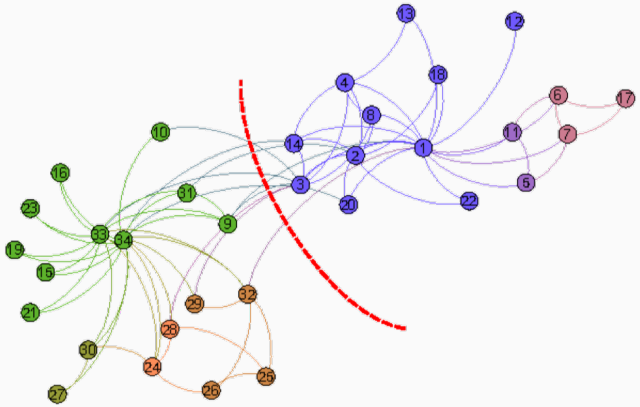
$$\min_S \textbf{cut}(S, V \setminus S) \quad \text{such that} \quad \textbf{min}\left(|S|, |V \setminus S|\right) \geq \beta \cdot n \text{ for } \beta \leq .5$$

Last class we focused on the extreme case where $\beta = 1/2$.

Basic spectral clustering method:

- Compute second smallest eigenvector of graph, $\mathbf{v}_{n-1}$.
- $\mathbf{v}_{n-1}$ has an entry for every node i in the graph.
- If the i^{th} entry is positive, put node i in T.
- Otherwise if the i^{th} entry is negative, put i in S.

THE LAPLACIAN VIEW



(a) Zachary Karate Club Graph

For a cut indicator vector $\mathbf{c} \in \{-1, 1\}^n$ with $\mathbf{c}(i) = -1$ for $i \in S$ and $\mathbf{c}(i) = 1$ for $i \in T$:

- $\mathbf{c}^T L \mathbf{c} = 4 \cdot cut(S, T)$.
- $\mathbf{c}^T \mathbf{1} = |T| - |S|$.

Want to minimize both $\mathbf{c}^T L \mathbf{c}$ (cut size) and $|\mathbf{c}^T \mathbf{1}|$ (imbalance).

Perfectly balanced balanced cut problem:

$$\min_{\mathbf{c} \in \{-\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}\}^n} \mathbf{c}^T \mathbf{L} \mathbf{c} \text{ such that } \mathbf{c}^T \mathbf{1} = 0.$$

Relaxed perfectly balanced balanced cut problem:

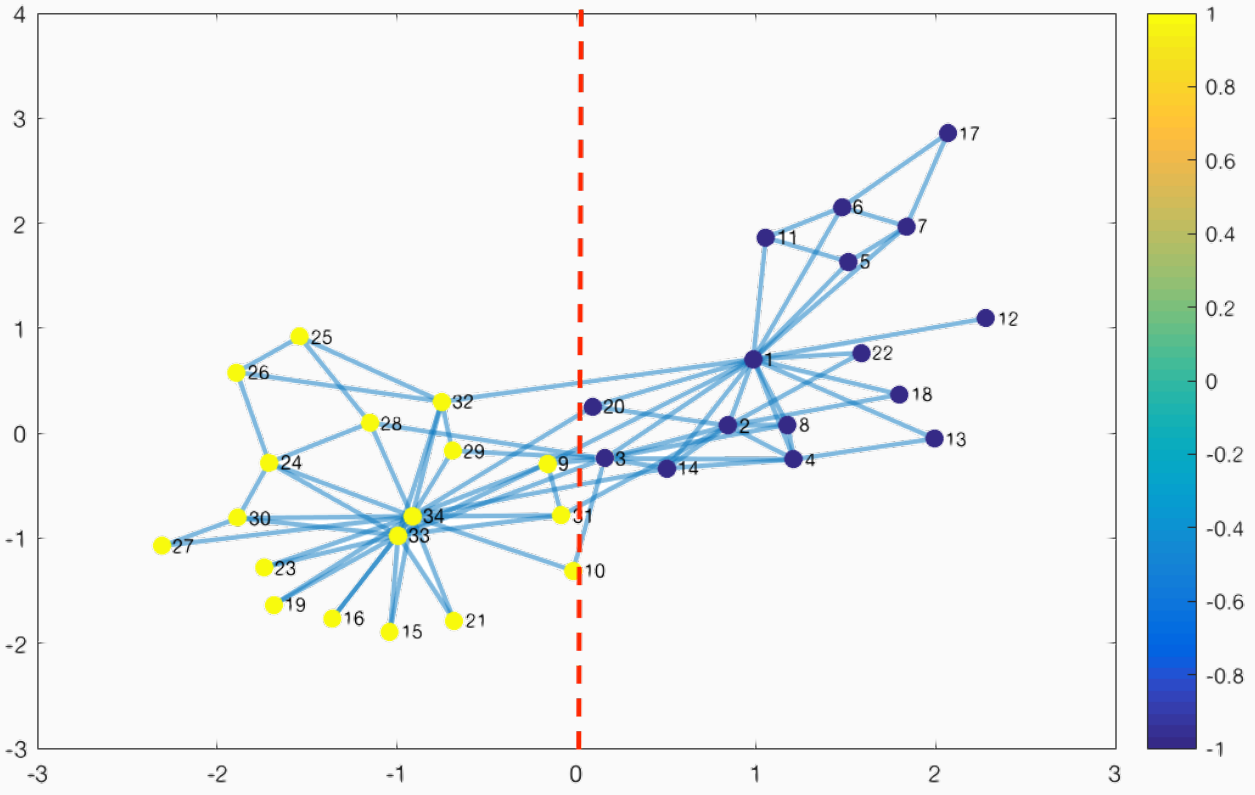
$$\min_{\|\mathbf{c}\|_2 = 1} \mathbf{c}^T \mathbf{L} \mathbf{c} \text{ such that } \mathbf{c}^T \mathbf{1} = 0.$$

Main result: The relaxed problem is exactly minimized by the second smallest eigenvector $\mathbf{v}_{n-1}$ of $\mathbf{L}$.

Final relax and round algorithm: Compute

$$\mathbf{v}_{n-1} = \underset{v \in \mathbb{R}^n \text{ with } \|\mathbf{v}\|=1,\ \mathbf{v}^T\mathbf{1}=0}{\arg\min} \mathbf{v}^T L \mathbf{v}$$

Set S to be all nodes with $\mathbf{v}_{n-1}(i) < 0$, and T to be all with $\mathbf{v}_{n-1}(i) \geq 0$. I.e. set $\mathbf{c} = \mathbf{sign}(\mathbf{v}_{n-1})$



GENERATIVE MODELS

So far: Showed that spectral clustering partitions a graph along a small cut between large pieces.

- No formal guarantee on the 'quality' of the partitioning.
- Can fail for worst case input graphs.

Common approach: Design a natural generative model that produces random but realistic inputs and analyze how the algorithm performs on inputs drawn from this model.

- Very common in algorithm design and analysis. Great way to start approaching a problem. Often our best way to understand why some algorithms "just work" in practice.
- Similar approach to Bayesian modeling in machine learning.

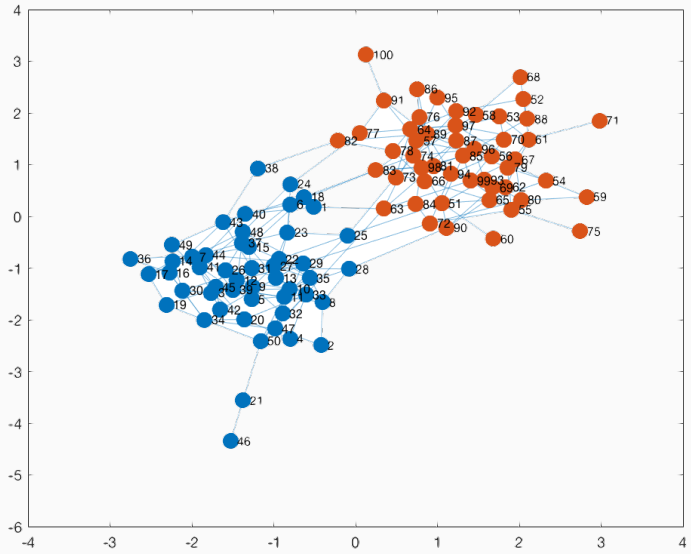
STOCHASTIC BLOCK MODEL

Ideas for a generative model for **social network graphs** that would allow us to understand partitioning?

Stochastic Block Model (Planted Partition Model):

Let $G_n(p, q)$ be a distribution over graphs on n nodes, split equally into two groups B and C, each with $n/2$ nodes.

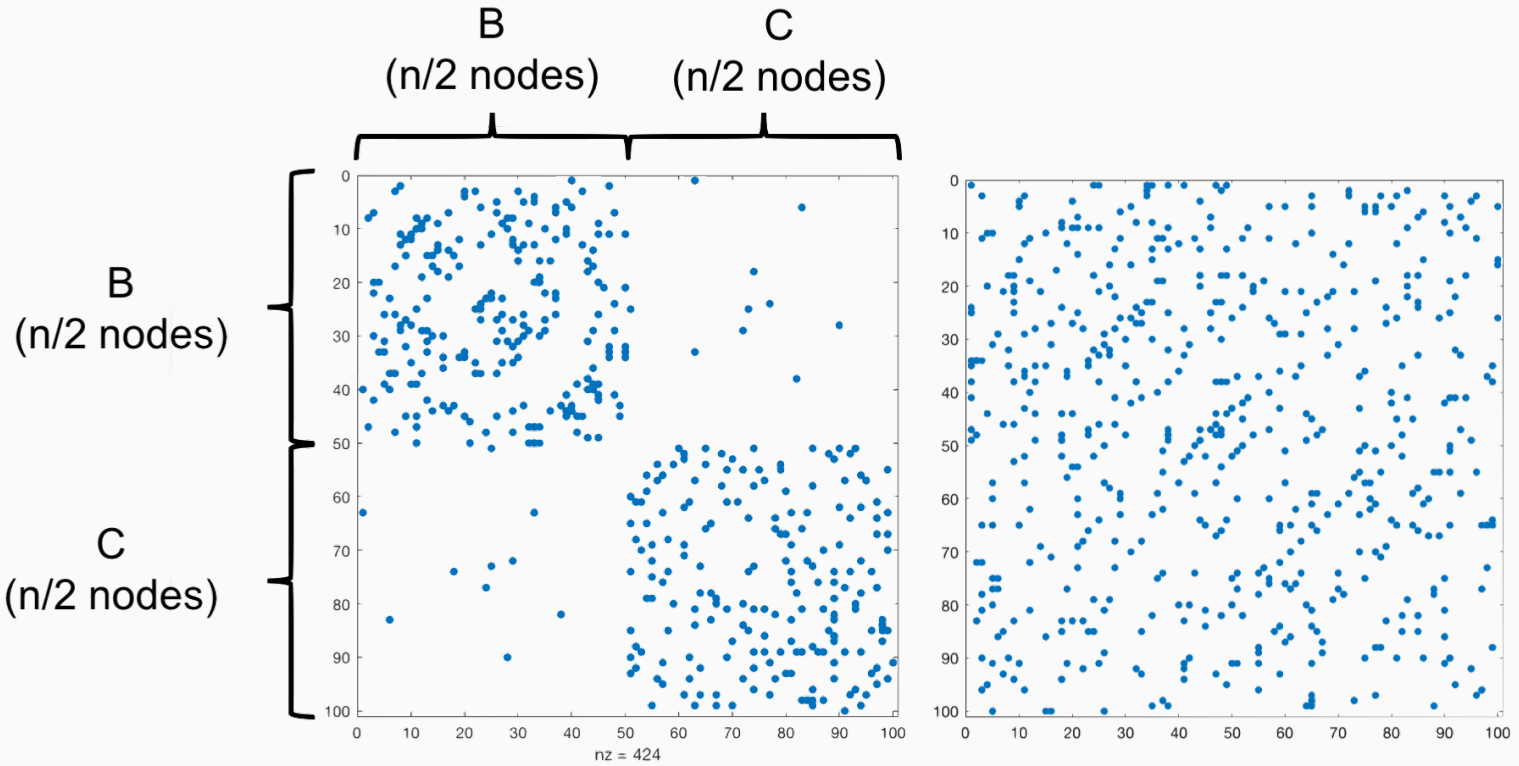
- Any two nodes in the same group are connected with probability p (including self-loops).
- Any two nodes in different groups are connected with prob. $q < p$.



LINEAR ALGEBRAIC VIEW

Let G be a stochastic block model graph drawn from $G_n(p, q)$.

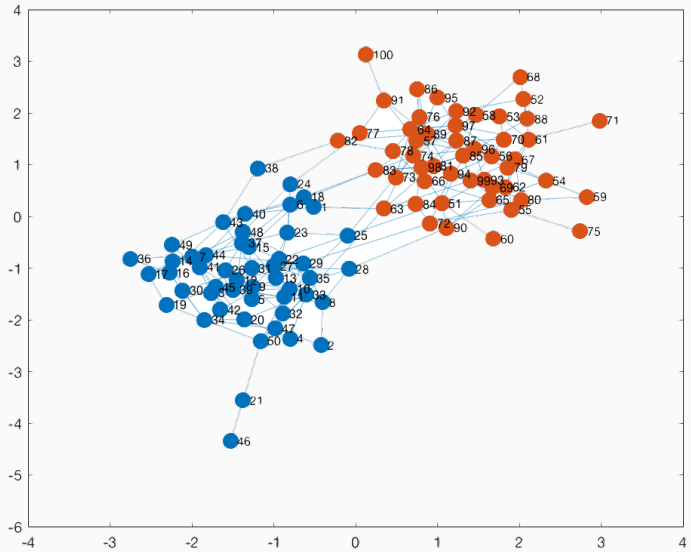
- Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ denote the adjacency matrix of G.



Note that we are arbitrarily ordering the nodes in A by group. In reality A would look "scrambled" as on the right.

STOCHASTIC BLOCK MODEL

Goal is to find the "ground truth" balanced partition B, C using our standard spectal method.

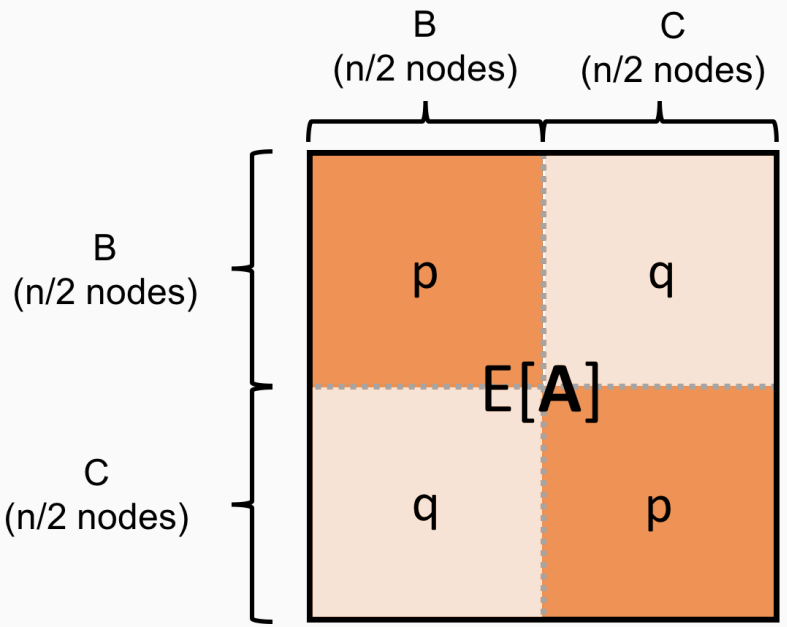


To do so, we need to understand the second smallest eigenvector of $\mathsf{L} = \mathsf{D} - \mathsf{A}$. We will start by considering the expected value of these matrices:

$$\mathbb{E}[\mathsf{L}] = \mathbb{E}[\mathsf{D}] - \mathbb{E}[\mathsf{A}].$$

EXPECTED ADJACENCY SPECTRUM

Letting G be a stochastic block model graph drawn from $G_n(p, q)$ and $\mathbf{A} \in \mathbb{R}^{n \times n}$ be its adjacency matrix. $(\mathbb{E}[\mathbf{A}])_{i,j} = p$ for i, j in same group, $(\mathbb{E}[\mathbf{A}])_{i,j} = q$ otherwise.



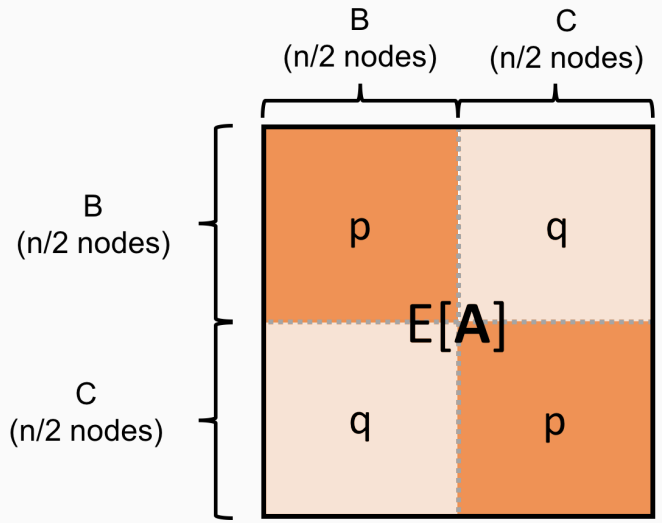
EXPECTED LAPLACIAN

What is the expected Laplacian of $G_n(p, q)$?

$\mathbb{E}[\mathsf{A}]$ and $\mathbb{E}[\mathsf{L}]$ have the same eigenvectors and eigenvalues are equal up to a shift/inversion. So second largest eigenvector of $\mathbb{E}[\mathsf{A}]$ is the same as the second smallest of $\mathbb{E}[\mathsf{L}]$

EXPECTED ADJACENCY SPECTRUM

Letting G be a stochastic block model graph drawn from $G_n(p, q)$ and $\mathbf{A} \in \mathbb{R}^{n \times n}$ be its adjacency matrix, what are the eigenvectors and eigenvalues of $\mathbb{E}[\mathbf{A}]$?



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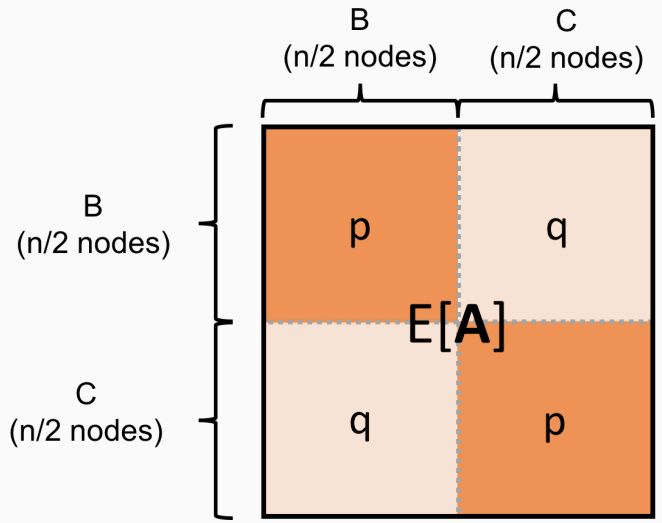
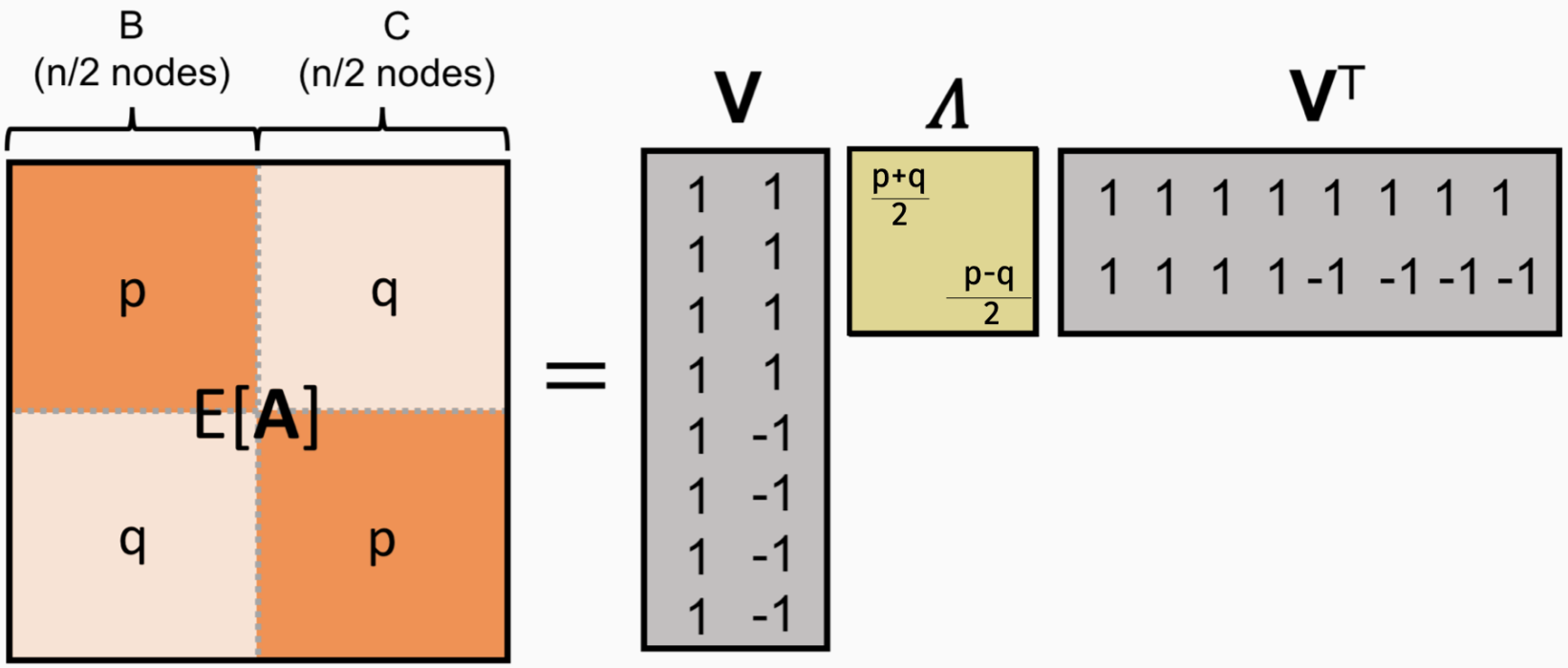


EXPECTED ADJACENCY SPECTRUM



- $\bar{\mathbf{v}}_1 \sim \mathbf{1}$ with eigenvalue $\lambda_1 = \frac{(p+q)n}{2}$.
- $\bar{\mathbf{v}}_2 \sim \boldsymbol{\chi}_{B,C}$ with eigenvalue $\lambda_2 = \frac{(p-q)n}{2}$.

If we compute $\bar{\mathbf{v}}_2$ then we exactly recover the communities B and C!

EXPECTED LAPLACIAN SPECTRUM

Upshot: The second smallest eigenvector of $\mathbb{E}[\mathbf{L}]$, equivalently the second largest of $\mathbb{E}[\mathbf{A}]$, is exactly $\boldsymbol{\chi}_{B,C}$ – the indicator vector for the cut between the communities.

- If the random graph G (equivilantly $\mathbf{A}$ and $\mathbf{L}$) were exactly equal to its expectation, partitioning using this eigenvector would exactly recover communities B and C.

How do we show that a matrix (e.g., $\mathbf{A}$) is close to its expectation? **Matrix concentration inequalities.**

- Analogous to scalar concentration inequalities like Markovs, Chebyshevs, Bernsteins.

Alon, Krivelevich, Vu, 2002:

Matrix Concentration Inequality: If $p \geq O\left(\frac{\log^4 n}{n}\right)$, then with high probability

$$\|\mathbf{A} - \mathbb{E}[\mathbf{A}]\|_2 \leq O(\sqrt{pn}).$$

where $\|\cdot\|_2$ is the matrix spectral norm (operator norm).

Recall that $\|\mathbf{X}\|_2 = \max_{z \in \mathbb{R}^n : \|z\|_2 = 1} \|\mathbf{X}z\|_2 = \sigma_1(\mathbf{X})$.

$\|\mathbf{A}\|_2$ is on the order of $O(pn)$ so another way of thinking about the right hand side is $\frac{\|\mathbf{A}\|_2}{\sqrt{np}}$. I.e. get's better with p and n.

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EIGENVECTOR PERTURBATION

For the stochastic block model application, we want to show that the second eigenvectors of $\mathbf{A}$ and $\mathbb{E}[\mathbf{A}]$ are close. How does this relate to their difference in spectral norm?

Davis-Kahan Eigenvector Perturbation Theorem: Suppose $\mathbf{A}, \bar{\mathbf{A}} \in \mathbb{R}^{d \times d}$ are symmetric with $\|\mathbf{A} - \bar{\mathbf{A}}\|_2 \leq \epsilon$ and eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and $\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2, \ldots, \bar{\mathbf{v}}_n$. Letting $\theta(\mathbf{v}_i, \bar{\mathbf{v}}_i)$ denote the angle between $\mathbf{v}_i$ and $\bar{\mathbf{v}}_i$, for all i:

$$\sin[\theta(\mathbf{v}_i, \bar{\mathbf{v}}_i)] \leq \frac{\epsilon}{\min_{j \neq i} |\lambda_i - \lambda_j|}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $\bar{\mathbf{A}}$.

We will apply with $\bar{\mathbf{A}} = \mathbb{E}[\mathbf{A}]$.

EIGENVECTOR PERTURBATION

$$\underset{\mathbf{A}}{\begin{bmatrix} 1+\varepsilon & 0 \\ 0 & 1 \end{bmatrix}} - \underset{\bar{\mathbf{A}}}{\begin{bmatrix} 1 & 0 \\ 0 & 1+\varepsilon \end{bmatrix}} = \underset{\mathbf{A}-\bar{\mathbf{A}}}{\begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{bmatrix}}$$

Claim 1 (Matrix Concentration): For $p \geq O\left(\frac{\log^4 n}{n}\right)$,

$$\|A - \mathbb{E}[A]\|_2 \leq O(\sqrt{pn}).$$

Recall: $\mathbb{E}[A]$, has eigenvalues $\lambda_1 = \frac{(p+q)n}{2}$, $\lambda_2 = \frac{(p-q)n}{2}$, $\lambda_i = 0$ for $i \geq 3$.

$$\min_{j \neq i} |\lambda_i - \lambda_j| = \min\left(qn, \frac{(p-q)n}{2}\right).$$

Assume $\frac{(p-q)n}{2}$ will be the minimum of these two gaps.

Claim 2 (Davis-Kahan): For $p \geq O\left(\frac{\log^4 n}{n}\right)$,

$$\sin\theta(v_2, \bar{v}_2) \leq \frac{O(\sqrt{pn})}{\min_{j \neq i} |\lambda_i - \lambda_j|} \leq \frac{O(\sqrt{pn})}{(p-q)n/2} = O\left(\frac{\sqrt{p}}{(p-q)\sqrt{n}}\right)$$

(A slightly trickier analysis can remove the qn term entirely.)

APPLICATION TO STOCHASTIC BLOCK MODEL

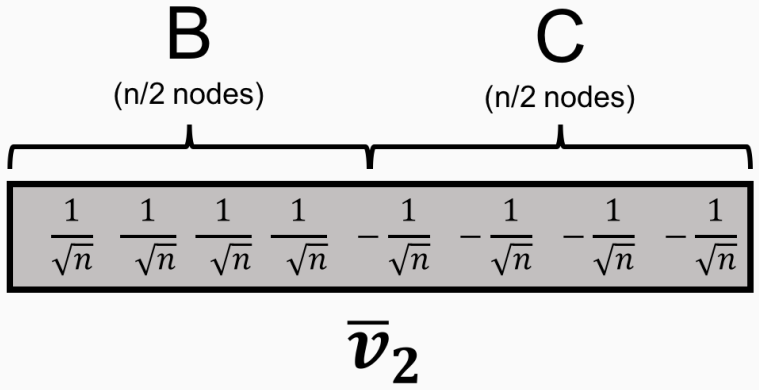
So far: $\sin\theta(\mathbf{v}_2, \bar{\mathbf{v}}_2) \leq O\left(\frac{\sqrt{p}}{(p-q)\sqrt{n}}\right)$. What does this give us?

- Can show that this implies $\|\mathbf{v}_2 - \bar{\mathbf{v}}_2\|_2^2 \leq O\left(\frac{p}{(p-q)^2 n}\right)$.

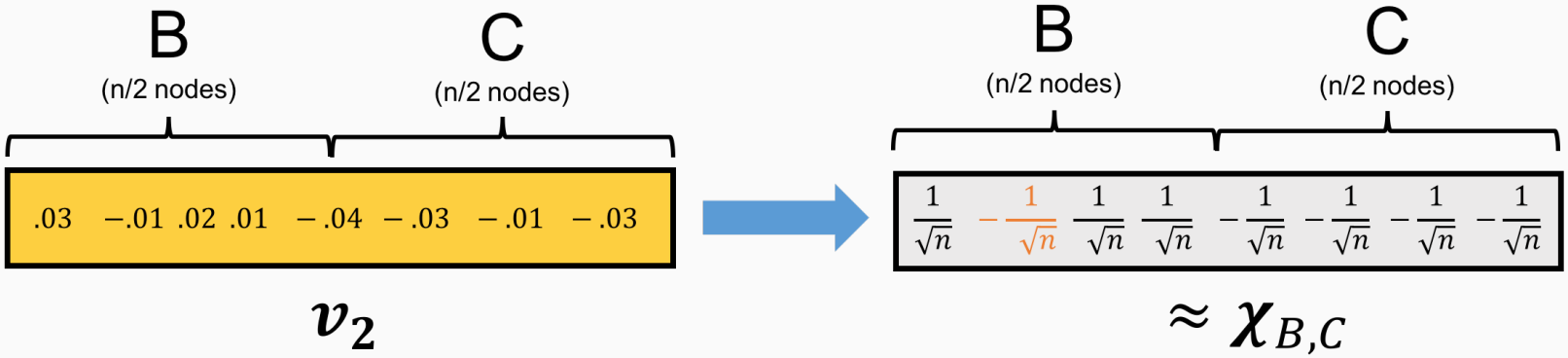
APPLICATION TO STOCHASTIC BLOCK MODEL

So far: $\|\mathbf{v}_2 - \bar{\mathbf{v}}_2\|_2^2 \leq O\left(\frac{p}{(p-q)^2 n}\right)$

- $\bar{\mathbf{v}}_2$ is $\frac{1}{\sqrt{n}}\chi_{B,C}$: the community indicator vector.



- We want to show that $\mathbf{sign}(\mathbf{v}_2)$ and $\bar{\mathbf{v}}_2$ are close. They only differ at locations where $\mathbf{v}_2$ and $\bar{\mathbf{v}}_2$ differ in sign.



Main argument:

- Every i where $v_2(i), \bar{v}_2(i)$ differ in sign contributes $\geq \frac{1}{n}$ to $\|\mathbf{v}_2 - \bar{\mathbf{v}}_2\|_2^2$.
- We know that $\|\mathbf{v}_2 - \bar{\mathbf{v}}_2\|_2^2 \leq O\left(\frac{p}{(p-q)^2 n}\right)$.
- So $\mathbf{v}_2$ and $\bar{\mathbf{v}}_2$ differ in sign in at most $O\left(\frac{p}{(p-q)^2}\right)$ positions.

Upshot: If G is a stochastic block model graph with adjacency matrix $\mathbf{A}$, if we compute its second largest eigenvector $\mathbf{v}_2$ and assign nodes to communities according to the sign pattern of this vector, we will correctly assign all but $O\left(\frac{p}{(p-q)^2}\right)$ nodes.

- **Hard case:** Suppose $q = .8p$ so $\frac{p}{(p-q)^2} = 25/p$. Even if p is really small, i.e. $p = 250/n$, then we assign roughly 90% of nodes to the right partition.

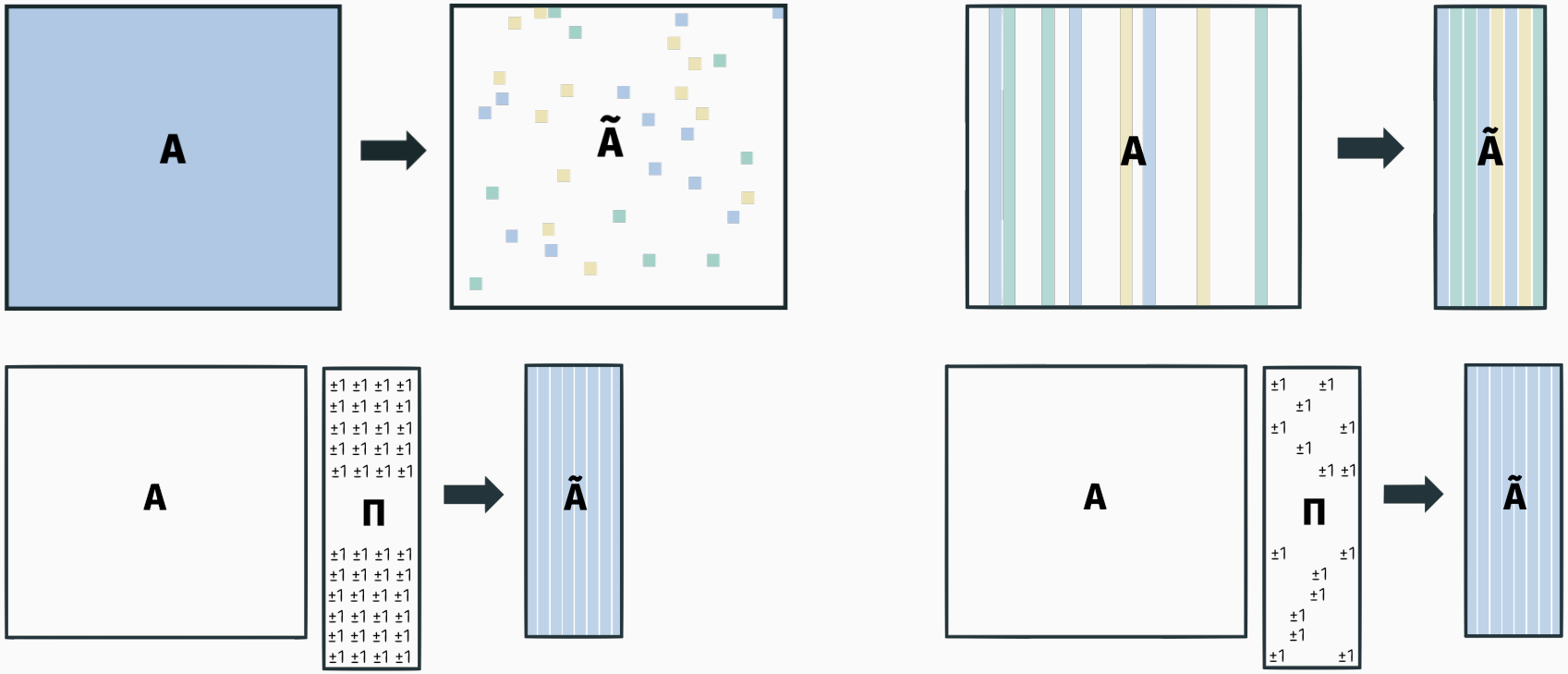
Forget about the previous problem, but still consider the matrix $\mathbf{M} = \mathbb{E}[\mathbf{A}]$.

- Dense $n \times n$ matrix.
- Computing top eigenvectors takes $\approx O(n^2/\sqrt{\epsilon})$ time.

If someone asked you to speed this up and return approximate top eigenvectors, what could you do?

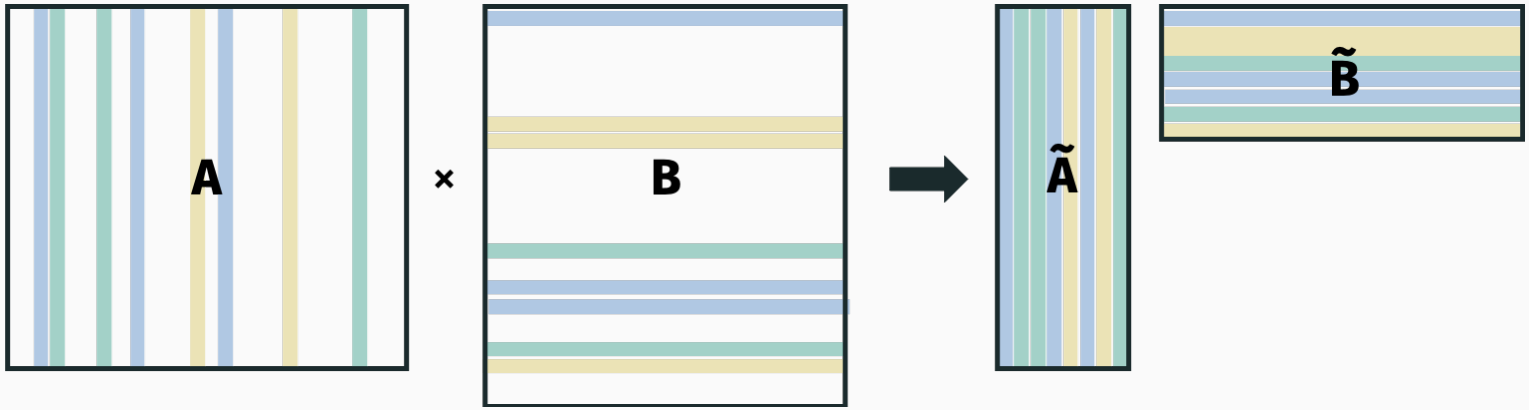
Main idea: If you want to compute singular vectors, multiply two matrices, solve a regression problem, etc.:

1. Compress your matrices using a randomized method (e.g. subsampling).
2. Solve the problem on the smaller or sparser matrix.
   - $\tilde{\mathbf{A}}$ called a "sketch" or "coreset" for $\mathbf{A}$.

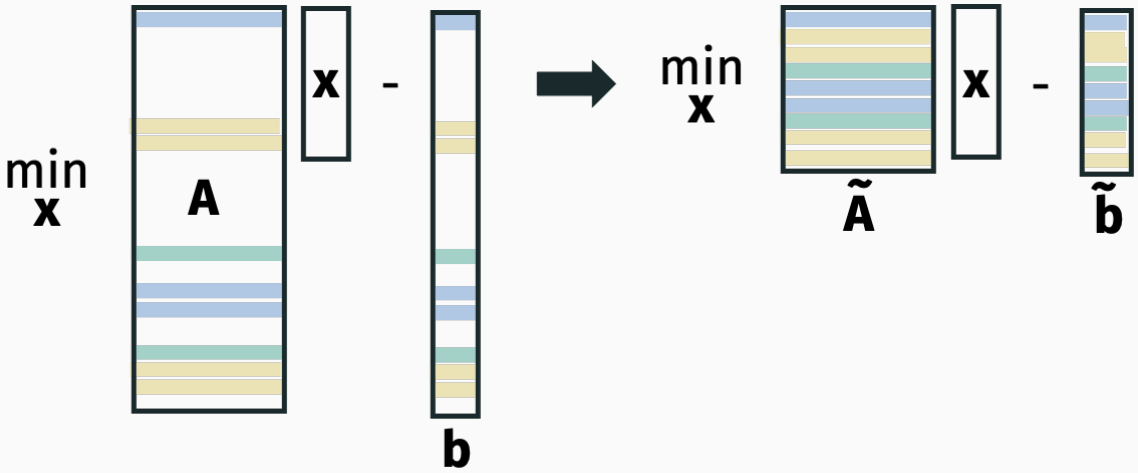


BREAK

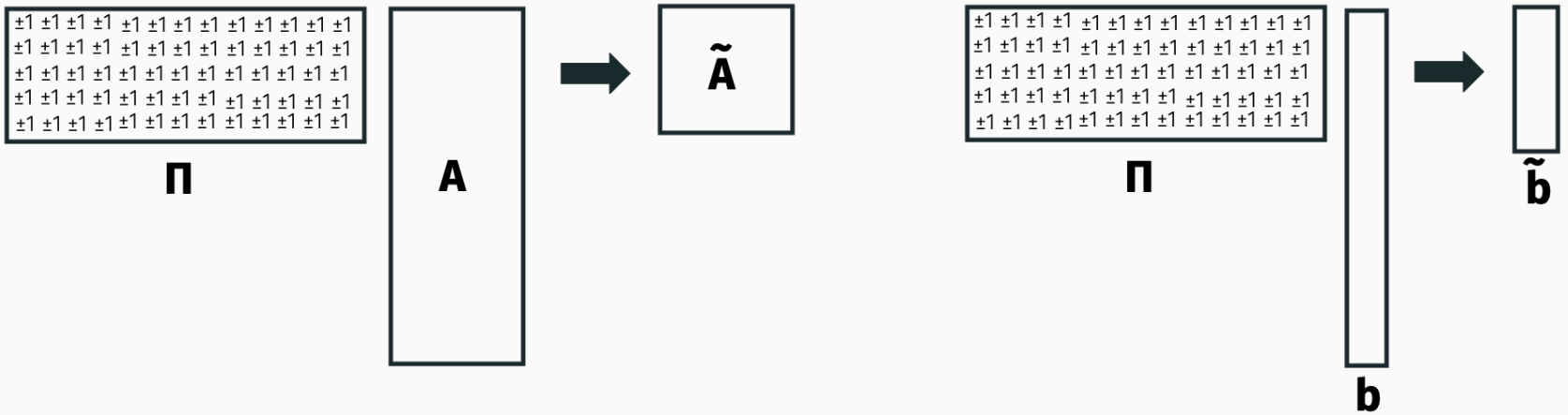
Approximate matrix multiplication:



Approximate regression:



Today's example: Randomized approximate regression using a Johnson-Lindenstrauss matrix for compression.



Input: $\mathbf{A} \in \mathbb{R}^{n \times d}$, $\mathbf{b} \in \mathbb{R}^n$.

Goal: Let $\mathbf{x}^* = \arg\min_{\mathbf{x}} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2$. Let $\tilde{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{\Pi}\mathbf{A}\mathbf{x} - \mathbf{\Pi}\tilde{\mathbf{b}}\|_2^2$

$$\text{Want: } \|\mathbf{A}\tilde{\mathbf{x}} - \mathbf{b}\|_2^2 \leq (1+\epsilon)\|\mathbf{A}\mathbf{x}^* - \mathbf{b}\|_2^2$$

Theorem (Randomized Linear Regression)

Let $\mathbf{\Pi}$ *be a JL matrix (random Gaussian, sign, sparse random, etc.) with* $m = O\left(\frac{d}{\epsilon^2}\right)$ *rows*[1]. *Then with probability* 9/10, *for any* $\mathbf{A} \in \mathbb{R}^{n \times d}$ *and* $\mathbf{b} \in \mathbb{R}^n$,

$$\|\mathbf{A}\tilde{\mathbf{x}} - \mathbf{b}\|_2^2 \leq (1+\epsilon)\|\mathbf{A}\mathbf{x}^* - \mathbf{b}\|_2^2$$

where $\tilde{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{\Pi}\mathbf{A}\mathbf{x} - \mathbf{\Pi}\mathbf{b}\|_2^2$.

[1] This can be improved to $O(d/\epsilon)$ with a tighter analysis

PLAN

- Prove this theorem using an ϵ-net argument, which is a popular technique for applying our standard concentration inequality + union bound argument to an infinite number of events.
- These sort of arguments appear all the time in theoretical algorithms and ML research, so this part of lecture is as much about the technique as the final result.

Claim: Suffices to prove that for all $\mathbf{x} \in \mathbb{R}^d$,

$$(1-\epsilon)\|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2^2 \leq \|\boldsymbol{\Pi}\mathbf{A}\mathbf{x}-\boldsymbol{\Pi}\mathbf{b}\|_2^2 \leq (1+\epsilon)\|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2^2$$

Lemma (Distributional JL)

If $\mathbf{\Pi}$ *is chosen to a random Gaussian matrix, sign matrix, sparse random matrix, etc. (scaled by* $1/\sqrt{m}$*) with* $m = O\left(\frac{\log(1/\delta}{\epsilon^2}\right)$ *rows then for any fixed* $\mathbf{y}$*,*

$$(1-\epsilon)\|\mathbf{y}\|_2^2 \leq \|\mathbf{\Pi y}\|_2^2 \leq (1+\epsilon)\|\mathbf{y}\|_2^2$$

with probability $(1-\delta)$*.*

Corollary: For any fixed $\mathbf{x}$, with probability $(1-\delta)$,

$$(1-\epsilon)\|\mathbf{Ax}-\mathbf{b}\|_2^2 \leq \|\mathbf{\Pi Ax}-\mathbf{\Pi b}\|_2^2 \leq (1+\epsilon)\|\mathbf{Ax}-\mathbf{b}\|_2^2.$$

FOR ANY TO FOR ALL

How do we go from "for any fixed $\mathbf{x}$" to "for all $\mathbf{x} \in \mathbb{R}^d$".

This statement requires establishing a Johnson-Lindenstrauss type bound for an infinity of possible vectors $(\mathbf{Ax} - \mathbf{b})$, which can't be tackled directly with a union bound argument.

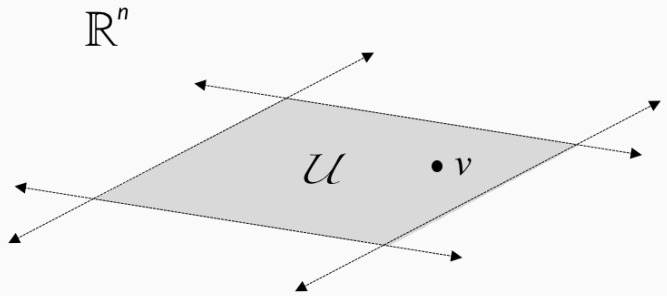
Note that all vectors of the form $(\mathbf{Ax} - \mathbf{b})$ lie in a low dimensional subspace: spanned by $d + 1$ vectors, where d is the width of $\mathbf{A}$. So even though the set is infinite, it is "simple" in some way. Parameterized by just $d + 1$ numbers.

Theorem (Subspace Embedding from JL)

Let $\mathcal{U} \subset \mathbb{R}^n$ be a d-dimensional linear subspace in $\mathbb{R}^n$. If $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$ is chosen from any distribution $\mathcal{D}$ satisfying the Distributional JL Lemma, then with probability $1 - \delta$,

$$(1-\epsilon)\|\mathbf{v}\|_2^2 \leq \|\mathbf{\Pi}\mathbf{v}\|_2^2 \leq (1+\epsilon)\|\mathbf{v}\|_2^2$$

for <u>all</u> $\mathbf{v} \in \mathcal{U}$, as long as $m = O\left(\frac{d\log(1/\epsilon)+\log(1/\delta)}{\epsilon^2}\right)$.[2]



[2] It's possible to obtain a slightly tighter bound of $O\left(\frac{d+\log(1/\delta)}{\epsilon^2}\right)$. It's a nice challenge to try proving this.

Corollary: If we choose $\mathbf{\Pi}$ and properly scale, then with $O\left(d/\epsilon^2\right)$ rows,

$$(1-\epsilon)\|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2^2 \leq \|\mathbf{\Pi}\mathbf{A}\mathbf{x}-\mathbf{\Pi}\mathbf{b}\|_2^2 \leq (1+\epsilon)\|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2^2$$

for all $\mathbf{x}$ and thus

$$\|\mathbf{A}\tilde{\mathbf{x}}-\mathbf{b}\|_2^2 \leq (1+O(\epsilon)) \min_{\mathbf{x}} \|\mathbf{A}\mathbf{x}-\mathbf{b}\|_2^2.$$

I.e., our main theorem is proven.

Proof: Apply Subspace Embedding Thm. to the $(d+1)$ dimensional subspace spanned by $\mathbf{A}$'s d columns and $\mathbf{b}$. Every vector $\mathbf{A}\mathbf{x}-\mathbf{b}$ lies in this subspace.

Theorem (Subspace Embedding from JL)

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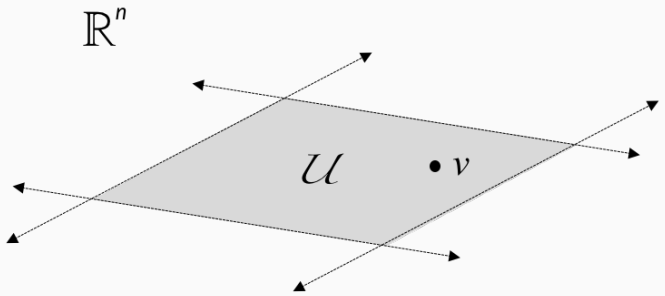
$$(1-\epsilon)\|\mathbf{v}\|_2^2 \leq \|\mathbf{\Pi}\mathbf{v}\|_2^2 \leq (1+\epsilon)\|\mathbf{v}\|_2^2 \tag{1}$$

for <u>all</u> $\mathbf{v} \in \mathcal{U}$, as long as $m = O\left(\frac{d\log(1/\epsilon)+\log(1/\delta)}{\epsilon^2}\right)$



Subspace embeddings have tons of other applications!

$$(1-\epsilon)\|\mathbf{v}\|_2^2 \leq \|\Pi\mathbf{v}\|_2^2 \leq (1+\epsilon)\|\mathbf{v}\|_2^2 \tag{2}$$

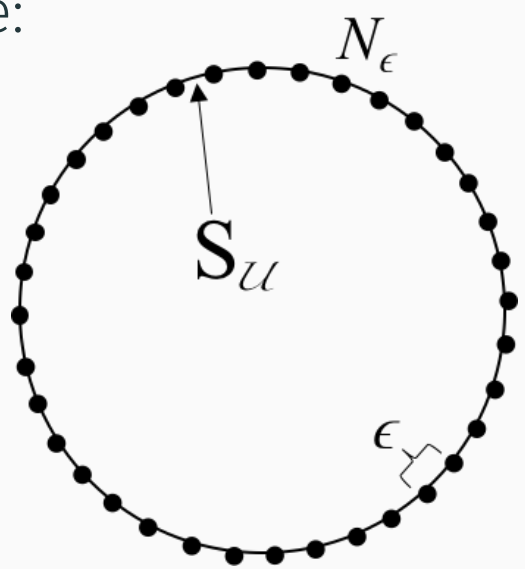
First Observation: The theorem holds as long as (1) holds for all $\mathbf{w}$ on the unit sphere in $\mathcal{U}$. Denote the sphere $S_{\mathcal{U}}$:

$$S_{\mathcal{U}} = \{\mathbf{w} \mid \mathbf{w} \in \mathcal{U} \text{ and } \|\mathbf{w}\|_2 = 1\}.$$

Follows from linearity: Any point $\mathbf{v} \in \mathcal{U}$ can be written as $c\mathbf{w}$ for some scalar c and some point $\mathbf{w} \in S_{\mathcal{U}}$.

- If $(1-\epsilon)\|\mathbf{w}\|_2 \leq \|\mathbf{\Pi}\mathbf{w}\|_2 \leq (1+\epsilon)\|\mathbf{w}\|_2$.
- then $c(1-\epsilon)\|\mathbf{w}\|_2 \leq c\|\mathbf{\Pi}\mathbf{w}\|_2 \leq c(1+\epsilon)\|\mathbf{w}\|_2$,
- and thus $(1-\epsilon)\|c\mathbf{w}\|_2 \leq \|\mathbf{\Pi}c\mathbf{w}\|_2 \leq (1+\epsilon)\|c\mathbf{w}\|_2$.

Intuition: There are not too many "different" points on a d-dimensional sphere:



Goal: Find a set N_ϵ such that, for every $\mathbf{v} \in S_{\mathcal{U}}$, there is some point $\mathbf{w} \in N_\epsilon$ such that $\|\mathbf{w} - \mathbf{v}\|_2 \leq \epsilon$. N_ϵ is called an "ϵ"-net. If we can prove

$$(1-\epsilon)\|\mathbf{w}\|_2 \leq \|\Pi\mathbf{w}\|_2 \leq (1+\epsilon)\|\mathbf{w}\|_2$$

for all points $\mathbf{w} \in N_\epsilon$, we can hopefully extend to all of $S_{\mathcal{U}}$.

ϵ-NET FOR THE SPHERE

Lemma (ϵ-net for the sphere)

For any $\epsilon \leq 1$, there exists a set $N_\epsilon \subset S_{\mathcal{U}}$ with $|N_\epsilon| = \left(\frac{3}{\epsilon}\right)^d$ such that $\forall \mathbf{v} \in S_{\mathcal{U}}$,

$$\min_{\mathbf{w} \in N_\epsilon} \|\mathbf{v} - \mathbf{w}\|_2 \leq \epsilon.$$

Take this claim to be true for now: we will prove later.

1. Preserving norms of all points in net N_ϵ.

Set $\delta' = \frac{1}{|\mathcal{N}_\epsilon|} \cdot \delta = \left(\frac{\epsilon}{3}\right)^d \cdot \delta$. As long as Π has $O\left(\frac{\log(1/\delta')}{\epsilon^2}\right)$ $= O\left(\frac{d\log(1/\epsilon)+\log(1/\delta)}{\epsilon^2}\right)$ rows, then by a union bound,

$$(1-\epsilon)\|\mathbf{w}\|_2 \leq \|\Pi\mathbf{w}\|_2 \leq (1+\epsilon)\|\mathbf{w}\|_2.$$

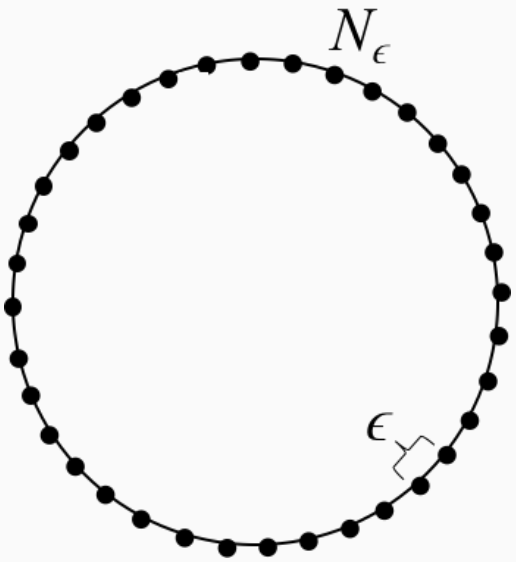
for all $\mathbf{w} \in N_\epsilon$,with probability $1-\delta$.

2. Extending to all points in the sphere.

For some $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2 \ldots \in N_\epsilon$, any $\mathbf{v} \in S_{\mathcal{U}}$ can be written:

$$\mathbf{v} = \mathbf{w}_0 + c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \ldots$$

for constants $c_1, c_2, \ldots$ where $|c_i| \leq \epsilon^i$.



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for constants $c_1, c_2, \ldots$ where $|c_i| \leq \epsilon^i$.

Greedy construction:

$$\mathbf{w}_0 = \min_{\mathbf{w} \in \mathcal{N}_\epsilon} \|\mathbf{v} - \mathbf{w}_0\|_2 \qquad \mathbf{r}_0 = \mathbf{v} - \mathbf{w}_0$$

$$\mathbf{w}_1 = \min_{\mathbf{w} \in \mathcal{N}_\epsilon} \left\| \frac{\mathbf{r}_0}{\|\mathbf{r}_0\|} - \mathbf{w}_0 \right\|_2 \quad c_1 = \|\mathbf{r}_0\|_2 \quad \mathbf{r}_1 = \mathbf{v} - \mathbf{w}_0 - c_1\mathbf{w}_1$$

$$\mathbf{w}_2 = \min_{\mathbf{w} \in \mathcal{N}_\epsilon} \left\| \frac{\mathbf{r}_1}{\|\mathbf{r}_1\|} - \mathbf{w}_0 \right\|_2 \quad c_2 = \|\mathbf{r}_1\|_2 \quad \mathbf{r}_2 = \mathbf{v} - \mathbf{w}_0 - c_1\mathbf{w}_1 - c_2\mathbf{w}_2$$

$$\vdots$$

2. Extending to all points in the sphere.

Applying triangle inequality, we have that:

$$\begin{aligned}
\|\mathbf{\Pi v}\|_2 &= \|\mathbf{\Pi w}_0 + c_1\mathbf{\Pi w}_1 + c_2\mathbf{\Pi w}_2 + \dots\| \\
&\leq \|\mathbf{\Pi w}_0\| + c_1\|\mathbf{\Pi w}_1\| + c_2\|\mathbf{\Pi w}_2\| + \dots \\
&\leq \|\mathbf{\Pi w}_0\| + \epsilon\|\mathbf{\Pi w}_1\| + \epsilon^2\|\mathbf{\Pi w}_2\| + \dots \\
&\leq (1+\epsilon) + \epsilon(1+\epsilon) + \epsilon^2(1+\epsilon) + \dots \\
&\leq 1 + 4\epsilon.
\end{aligned}$$

3. Preserving norm of v.

Similarly,

$$\begin{aligned}
\|\mathbf{\Pi v}\|_2 &= \|\mathbf{\Pi} w_0 + c_1 \mathbf{\Pi} w_1 + c_2 \mathbf{\Pi} w_2 + \dots\| \\
&\geq \|\mathbf{\Pi} w_0\| - \epsilon\|\mathbf{\Pi} w_1\| - \epsilon^2\|\mathbf{\Pi} w_2\| - \dots \\
&\geq (1-\epsilon) - \epsilon(1+\epsilon) - \epsilon^2(1+\epsilon) - \dots \\
&\geq 1 - 4\epsilon.
\end{aligned}$$

So we have proven

$$(1 - O(\epsilon)) \|\mathbf{v}\|_2 \leq \|\mathbf{\Pi v}\|_2 \leq (1 + O(\epsilon)) \|\mathbf{v}\|_2$$

for all $\mathbf{v} \in S_{\mathcal{U}}$, which in turn implies,

$$(1 - O(\epsilon)) \|\mathbf{v}\|_2^2 \leq \|\mathbf{\Pi v}\|_2^2 \leq (1 + O(\epsilon)) \|\mathbf{v}\|_2^2$$

Adjusting ϵ proves the Subspace Embedding theorem.

Theorem (Subspace Embedding from JL)

Let $\mathcal{U} \subset \mathbb{R}^n$ be a d-dimensional linear subspace in $\mathbb{R}^n$. If $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$ is chosen from any distribution $\mathcal{D}$ satisfying the Distributional JL Lemma, then with probability $1 - \delta$,

$$(1 - \epsilon)\|\mathbf{v}\|_2^2 \leq \|\mathbf{\Pi}\mathbf{v}\|_2^2 \leq (1 + \epsilon)\|\mathbf{v}\|_2^2 \tag{3}$$

for <u>all</u> $\mathbf{v} \in \mathcal{U}$, as long as $m = O\left(\frac{d \log(1/\epsilon) + \log(1/\delta)}{\epsilon^2}\right)$

Subspace embeddings have many other applications!

For example, if $m = O(k/\epsilon)$, $\mathbf{\Pi}\mathbf{A}$ can be used to compute an approximate partial SVD, which leads to a $(1 + \epsilon)$ approximate low-rank approximation for $\mathbf{A}$.

ϵ-NET FOR THE SPHERE

Lemma (ϵ-net for the sphere)

For any $\epsilon \leq 1$, there exists a set $N_\epsilon \subset S_{\mathcal{U}}$ with $|N_\epsilon| = \left(\frac{3}{\epsilon}\right)^d$ such that $\forall \mathbf{v} \in S_{\mathcal{U}}$,

$$\min_{\mathbf{w} \in N_\epsilon} \|\mathbf{v} - \mathbf{w}\| \leq \epsilon.$$

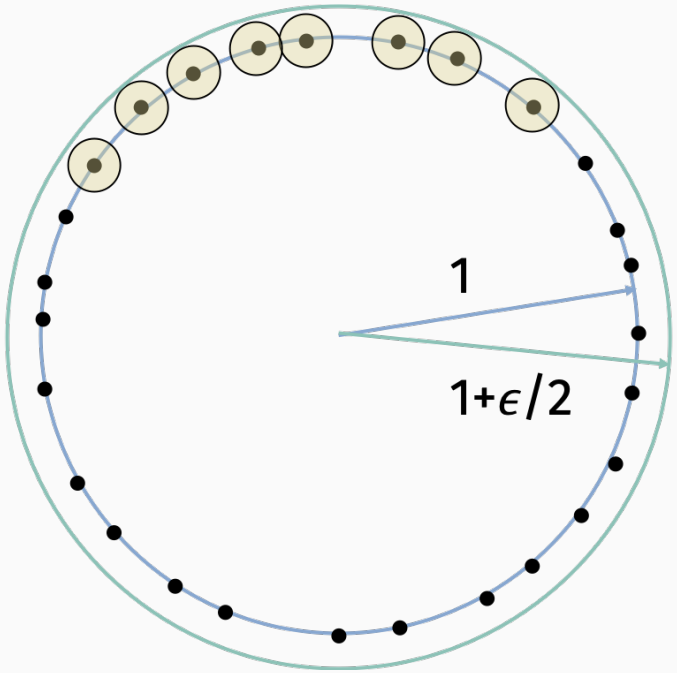
Imaginary algorithm for constructing N_ϵ:

- Set $N_\epsilon = \{\}$
- While such a point exists, choose an arbitrary point $\mathbf{v} \in S_{\mathcal{U}}$ where $\nexists \mathbf{w} \in N_\epsilon$ with $\|\mathbf{v} - \mathbf{w}\| \leq \epsilon$. Set $N_\epsilon = N_\epsilon \cup \{\mathbf{w}\}$.

After running this procedure, we have $N_\epsilon = \{\mathbf{w}_1, \ldots, \mathbf{w}_{|N_\epsilon|}\}$ and $\min_{\mathbf{w} \in N_\epsilon} \|\mathbf{v} - \mathbf{w}\| \leq \epsilon$ for all $\mathbf{v} \in S_{\mathcal{U}}$ as desired.

ϵ-NET FOR THE SPHERE

How many steps does this procedure take?



Can place a ball of radius $\epsilon/2$ around each $\mathbf{w}_j$ without intersecting any other balls. All of these balls live in a ball of radius $1 + \epsilon/2$.

VOLUME ARGUMENT

Volume of d dimensional ball of radius r is

$$\text{vol}(d, r) = c \cdot r^d,$$

where c is a constant that depends on d, but not r. From previous slide we have:

$$\begin{aligned} \text{vol}(d, \epsilon/2) \cdot |N_\epsilon| &\leq \text{vol}(d, 1 + \epsilon/2) \\ |N_\epsilon| &\leq \frac{\text{vol}(d, 1 + \epsilon/2)}{\text{vol}(d, \epsilon/2)} \\ &\leq \left(\frac{1 + \epsilon/2}{\epsilon/2}\right)^d \leq \left(\frac{3}{\epsilon}\right)^d \end{aligned}$$

TIGHTER BOUND

You can actually show that $m = O\left(\frac{d+\log(1/\delta)}{\epsilon}\right)$ suffices to be a d dimensional subspace embedding, instead of the bound we proved of $m = O\left(\frac{d\log(1/\epsilon)+\log(1/\delta)}{\epsilon}\right)$.

The trick is to show that a constant factor net is actually all that you need instead of an ϵ factor.

RUNTIME CONSIDERATION

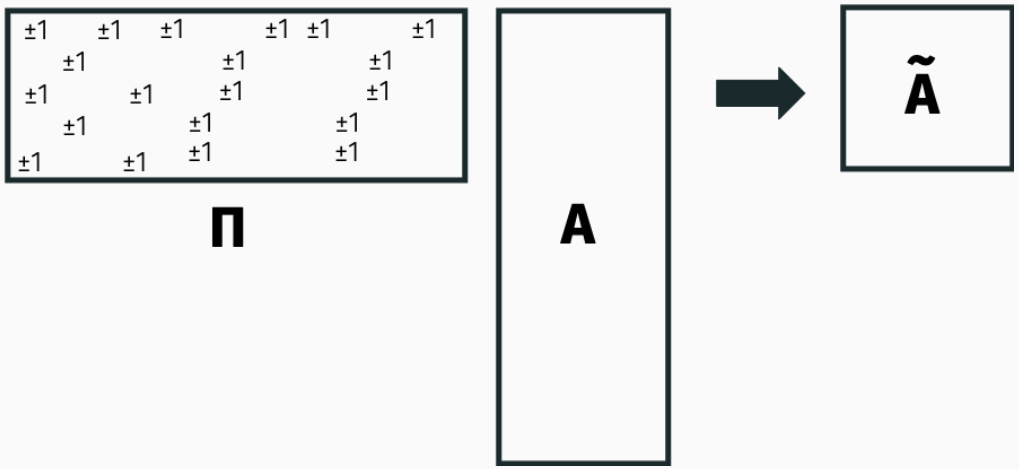
For $\epsilon, \delta = O(1)$, we need $\mathbf{\Pi}$ to have $m = O(d)$ rows.

- Cost to solve $\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2$:
  - $O(nd^2)$ time for direct method. Need to compute $(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b}$.
  - $O(nd) \cdot (\#\text{ of iterations})$ time for iterative method (GD, AGD, conjugate gradient method).
- Cost to solve $\|\mathbf{\Pi}\mathbf{A}\mathbf{x} - \mathbf{\Pi}\mathbf{b}\|_2^2$:
  - $O(d^3)$ time for direct method.
  - $O(d^2) \cdot (\#\text{ of iterations})$ time for iterative method.

RUNTIME CONSIDERATION

But time to compute $\mathbf{\Pi A}$ is an $(m \times n) \times (n \times d)$ matrix multiply: $O(mnd) = O(nd^2)$ time!

Goal: Develop faster Johnson-Lindenstrauss projections.



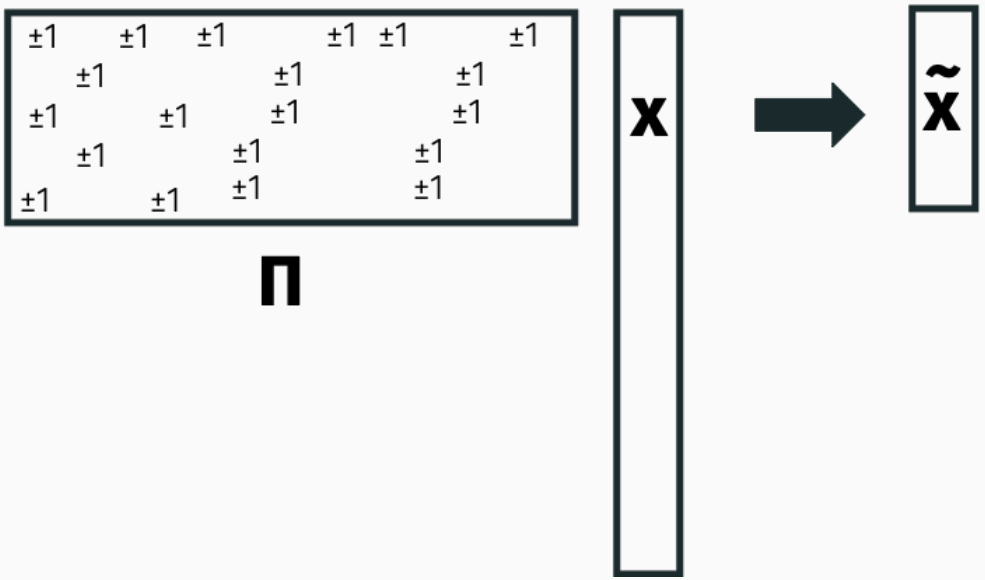
Typically using sparse and structured matrices.

Next class: We will describe a construction where $\mathbf{\Pi A}$ can be computed in $O(nd \log n)$ time.

RETURN TO SINGLE VECTOR PROBLEM

Goal: Develop methods that reduce a vector $\mathbf{x} \in \mathbb{R}^n$ down to $m \approx \frac{\log(1/\delta)}{\epsilon^2}$ dimensions in $o(mn)$ time and guarantee:

$$(1-\epsilon)\|\mathbf{x}\|_2^2 \leq \|\mathbf{\Pi x}\|_2^2 \leq (1+\epsilon)\|\mathbf{x}\|_2^2$$



There is a truly brilliant method that runs in $O(n \log n)$ time.

Preview: Will involve Fast Fourier Transform in disguise.