NYU TANDON SCHOOL OF ENGINEERING

New York University Tandon School of Engineering
Computer Science and Engineering

CS-GY 6763: Homework 4.
Due Monday, April 14th, 2025, 11:59pm.

Collaboration is allowed on this problem set, but solutions must be written-up individually. Please list collaborators for each problem separately, or write "No Collaborators" if you worked alone.

Problem 1: Concentration of Random Vectors

(12 pts) In Stochastic Gradient Descent, we replace the true gradient vector with a stochastic gradient that is equal to the true gradient in expectation. Our analysis in class *only used* equality in expectation, although more refined analysis of SGD often requires understanding how well the stochastic gradient *concentrates* around its expectation. Previously, all concentration results we studied apply to random numbers. For this problem, you will prove a basic concentration inequality for random vectors.

In particular, let $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \mathbb{R}^d$ be i.i.d. random vectors in d dimensions (independent, drawn from the same distribution) with mean $\boldsymbol{\mu}$. I.e., $\mathbb{E}[\mathbf{x}_i] = \boldsymbol{\mu}$. Further suppose that $\mathbb{E}\left[\|\mathbf{x}_i - \boldsymbol{\mu}\|_2^2\right] = \sigma^2$. σ^2 is a natural generalization of "variance" to a random vector. Let $\mathbf{s} = \frac{1}{k}\sum_{i=1}^{k} \mathbf{x}_i$. Prove that if $k \geq O\left(\frac{1/\delta}{\epsilon^2}\right)$ then

$$\Pr\left[\|\mathbf{s} - \boldsymbol{\mu}\|_2 \geq \epsilon\sigma\right] \leq \delta.$$

Problem 2: Regularization

(10 pts) Regularization is a popular technique in machine learning. It is often used to improve final test error, but can also help speed-up optimization methods like gradient descent by improving the condition number of the function being regularized. In particular, let $f(\mathbf{x})$ be a differentiable function mapping a length d vector $\mathbf{x}$ to a scalar value. Let g be the function with added Euclidean regularization:

$$g(\mathbf{x}) = f(\mathbf{x}) + \lambda\|\mathbf{x}\|_2^2$$

Above $\lambda > 0$ is a non-negative constant that controls the amount of regularization. Suppose f is α_1-strongly convex and β_1-smooth, so has condition number β_1/α_1. Prove that g is also convex and its condition number less than or equal to that of f.

Problem 3: Gradient Descent with Decaying Step-size

(12 pts) In class we showed that gradient descent with step size $\eta = R/G\sqrt{T}$ converges to an ϵ approximate minimizer of a convex G-Lipschitz function in $T = R^2G^2/\epsilon^2$ steps if our starting point $\mathbf{x}^{(0)}$ satisfies $\|\mathbf{x}^{(0)} - \mathbf{x}^*\|_2 \leq R$. Choosing this step size requires knowing G, R and moreover T in advance, which might not be reasonable in a lot of settings. For example, when training machine learning models, we might not be able to estimate how long it will take to reach a point where test accuracy levels off. Instead, we want to be able to keep running the algorithm, achieving better and better accuracy as we do.

Here, we analyze a variant of gradient descent with a variable step size that avoids this limitation. In particular, consider running gradient descent with the update $\mathbf{x}^{(i+1)} = \mathbf{x}^{(i)} - \eta\nabla f(\mathbf{x}^{(i)})$, where

$$\eta = \frac{f(\mathbf{x}^{(i)}) - f(\mathbf{x}^*)}{\|\nabla f(\mathbf{x}^{(i)})\|_2^2}.$$

This step size requires knowledge of $f(\mathbf{x}^*)$, but not $\mathbf{x}^*$, which may be reasonable in some settings. Moreover, since it's just one parameter, grid search can be more easily used to "guess" $f(\mathbf{x}^*)$ than the three parameters G, R, T. More complex approaches can remove the need to know this value entirely.

Prove that, if we run gradient descent for $T = O(R^2G^2/\epsilon^2)$ steps using the step size above then $\hat{\mathbf{x}} = \min_{i \in 0,\ldots,T} f(\mathbf{x}^{(i)})$ satisfies $f(\hat{\mathbf{x}}) \leq f(\mathbf{x}^*) + \epsilon$. **Hint:** Prove that our distance from the optimum $\|\mathbf{x}^{(i)} - \mathbf{x}^*\|_2$ always decreases with this choice of step size, and the decrease is larger if our gap from the objective value $f(\mathbf{x}^{(i)}) - f(\mathbf{x}^*)$ is larger.

Problem 4: Separation Oracles

(12 pts) Describe efficient separation oracles for each of the following families of convex sets. Here, "efficient" means linear time plus $O(1)$ calls to any additional oracles provided to you.

(a) The set $A \cap B$, given separation oracles for A and B.

(b) The ℓ_1 ball: $\|\mathbf{x}\|_1 \leq 1$.

(c) Any convex set A, given a *projection oracle* for A. Recall that a projection oracle, given a point $\mathbf{x}$, returns

$$\text{Proj}_A(\mathbf{x}) = \arg\min_{y \in A} \|\mathbf{x} - \mathbf{y}\|_2.$$

Above you may wish to use the following fact that was stated but not formally proven in class: for any point $\mathbf{x}$, convex set A, and $\mathbf{z} \in A$, $\|\mathbf{z} - \text{Proj}_A(\mathbf{x})\|_2 \leq \|\mathbf{z} - \mathbf{x}\|_2$.

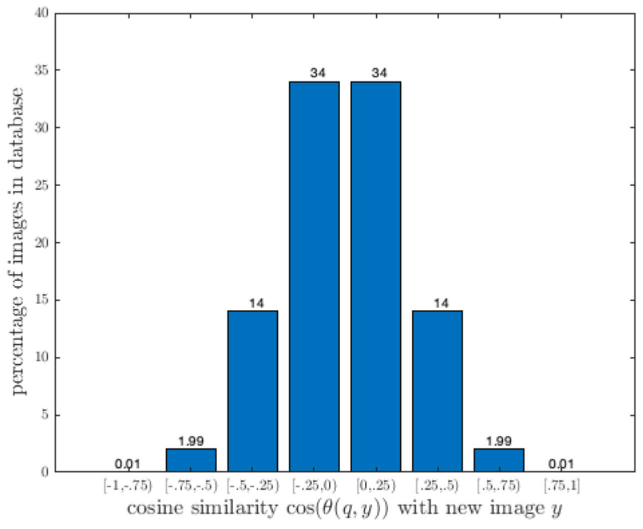
Problem 5: LSH in the Wild (Extra Credit)

This exercise does not involve formal proofs or analysis like more typical problem set problems. It will likely involve some coding or spreadsheet work.

(5 pts extra credit) To support its largely visual platform, Pinterest runs a massive image de-duplication operation built on Locality Sensitive Hashing for Cosine Similarity. You can read about the actual system here. All information and numbers below are otherwise purely hypothetical.

Pinterest has a database of $N = \mathbf{1}$ **billion** images. Each image in the database is pre-processed and represented as a vector $\mathbf{q} \in \mathbb{R}^d$. When a new image is pinned, it is also processed to form a vector $\mathbf{y} \in \mathbb{R}^d$. The goal is to check for any existing duplicates or near-duplicates to $\mathbf{y}$ in the database. Specifically, Pinterest would like to flag an image $\mathbf{q}$ as a near-duplicate to $\mathbf{y}$ if $\cos(\theta(\mathbf{q}, \mathbf{y})) \geq .98$. We want to find any near-duplicate with probability $\geq 99\%$.

Given this requirement, your job is to design a multi-table LSH scheme using SimHash to find candidate near-duplicates, which can then be checked directly against $\mathbf{y}$. To support this task, Pinterest has collected data on the empirical distribution of $\cos(\theta(\mathbf{q}, \mathbf{y}))$ for a typical new image $\mathbf{y}$. It roughly follows a bell-curve:



Pinterest wants to consider two possible computational targets for your LSH scheme, which will determine the speed of the de-duplication algorithm:

1. Ensure that no more than 1 million candidate near-duplicates are checked on average when a new image is pinned. "Checked" means that the image's cosine similarity with the new image is computed explicitly, which is a computationally expensive operation.

2. Ensure that no more than $200,000$ candidates are checked on average when a new image is pinned.

Based on the data above, describe how to set parameters for your LSH scheme to minimize the space (i.e., number of tables) used, while achieving each of the above goals. Justify your answers, and any assumptions you make. If you code anything up to help calculate your answer, please attach the code. As in lecture, you can assume that each hash table has $m = O(N)$ slots and this is large enough to ignore lower order terms depending on $1/m$.