

New York University Tandon School of Engineering
Computer Science and Engineering

CS-GY 9223I: Homework 4.

Due Monday, December 16th, 2019, 11:59pm.

Collaboration is allowed on this problem set, but solutions must be written-up individually. Please list collaborators for each problem separately, or write “No Collaborators” if you worked alone.

Problem 1: Restricted Isometry Property for JL Matrices

(15 pts) A k -sparse vector is any vector with k nonzero entries. Let \mathcal{S}_k be the set of all k -sparse vectors in \mathbb{R}^d . Show that, if $\Pi \in \mathbb{R}^{s \times d}$ is chosen to be a Johnson-Lindenstrauss embedding matrix (e.g. a scaled random Gaussian matrix, random sign matrix, etc.) with $s = O(\frac{k \log d}{\epsilon^2})$ rows then, with high probability,

$$(1 - \epsilon)\|x\|_2 \leq \|\Pi x\|_2 \leq (1 + \epsilon)\|x\|_2$$

for all $x \in \mathcal{S}_k$, simultaneously. You may use any of the results stated in class for JL matrices (except for the RIP property itself of course).

Problem 2: 18th Century Style Compressed Sensing

(15 pts) In class it was mentioned that there exist simpler compressed sensing schemes that work when noise/numerical precision is not an issue. Let $q_1, \dots, q_n \in \mathbb{R}$ be any set of *distinct* numbers. E.g. we could choose $[q_1, \dots, q_n] = [1, \dots, n]$. Consider the sensing matrix $A \in \mathbb{R}^{2k \times n}$:

$$A = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ q_1 & q_2 & q_3 & \dots & q_n \\ (q_1)^2 & (q_2)^2 & (q_3)^2 & \dots & (q_n)^2 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ (q_1)^{2k-1} & (q_2)^{2k-1} & (q_3)^{2k-1} & \dots & (q_n)^{2k-1} \end{bmatrix}$$

This A does not obey any sort of RIP property. Nevertheless, show that, if $x \in \mathbb{R}^n$ is a k sparse vector – i.e. $\|x\|_0 \leq k$ – then we can recover x from Ax . You don’t need to give an efficient algorithm. Just argue that for any given $y \in \mathbb{R}^{2k}$, there is at most one k -sparse x such that $y = Ax$. (Hint: Use that a non-zero degree p polynomial cannot have more than p roots. You may also want to use that the column and row rank of a matrix are always equal.)

Problem 3: Sparse Recovery for Dense Vectors – BONUS PROBLEM

(10 pts Bonus) A compressed sensing scheme typically recovers x from a linear sketch Ax whenever x is k -sparse. When x is not k -sparse, there is no guarantee about what is returned. E.g., for the measurement matrix A described above, for any specified k , there exists an algorithm $Decode(y)$ which returns x if $y = Ax$ for a k -sparse x . If $y \neq Ax$ for some k -sparse x , $Decode(y)$ can return anything. In this problem we consider an method that will still return *something useful* when x is not k -sparse.

In particular, your goal is to design a measurement matrix $B \in \mathbb{R}^{O(\log n) \times n}$ such that for *any* x (i.e. not necessarily sparse) it is possible to recover a single index/value pair (i, x_i) with $x_i \neq 0$ from Bx with constant probability (e.g. with success probability 9/10). Your algorithm can return *any* (i, x_i) as long as $x_i \neq 0$. **Hint: One possible B takes the form:**

$$B = \begin{bmatrix} AD_0 \\ AD_1 \\ AD_2 \\ \dots \\ AD_s \end{bmatrix}$$

where D_1, \dots, D_s are carefully (and randomly) constructed diagonal matrices and A is the matrix from Problem 2 with $k = O(1)$.

Problem 4: Communicating in the Dark is Easier with Shared Random Coins.

(10 pts) Suppose Jesse holds a subset of elements $J \subseteq \{1, \dots, n\}$. Leslie holds another subset $L \subseteq \{1, \dots, n\}$. Jesse and Leslie do not know what elements the other holds. Using as little communication as possible, Jesse wants to figure out if she or Leslie hold any unique elements – i.e. if there is any $j \in J \cup L - J \cap L$.

Show that, for some constant c , Leslie can send Jesse a single message of $O(\log^c n)$ bits that allows her to find such a j if one exists, with constant success probability.

You can assume that Jesse and Leslie decide on a strategy in advance, and that they have access to an unlimited source of shared random bits (e.g. that are published by some third party).

Hint: You might want to use the result from Problem 3. Even if you do not solve Problem 3, you can use the existence of a solution (e.g. a measurement matrix B and a recovery algorithm achieving the described goal.)

This result should surprise you! Even if Leslie *knew* all of Jesse's elements, $O(\log n)$ bits would be needed to communicate if they hold any unique elements. Here we are saying that nearly the same communication complexity can be achieved with *no prior knowledge* of J .